The Association
of
Engineering and Shipbuilding Draughtsmen.

AN INTRODUCTION TO ENGINEERING MATHEMATICS, WITH REFERENCE TO HEAVISIDE'S OPERATIONAL CALCULUS.

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Published by The Draughtsman Publishing Co., Ltd.,
96 St. George's Square, London, S.W.1.

SESSION 1947-48.

Printed by MILNE, TANNAHILL & METHVEN, LTD. (T.U.), 12-14 Mill Street, Perth.

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AN INTRODUCTION TO ENGINEERING MATHEMATICS, WITH REFERENCE TO HEAVISIDE'S OPERATIONAL CALCULUS.

(With Examples of an Electrical Nature)

BY

RONALD J. BIRKINSHAW, A.R.T.C.S., GRAD.I.E.E., GRAD.I.EC.E., M.J.INST.E.

INTRODUCTION.

The following section is merely a brief outline of the various methods used in the calculus.

For reasons of space-economy, the subject is not treated at great length, as many excellent text-books on this branch of mathematics, already exist.

A few of these books are mentioned in the reading list at the end of this pamphlet.

BRIEF RESUME OF THE CALCULUS.

Differentiation of y = f (x).

Let $y = x^3$

Imagine y given a small increase δy, and x a small increase δx.

$$\begin{aligned} \text{Thus } y + \delta y &= (x + \delta x)^3 \\ &= x^3 + 3x^2\,\delta x + 3x\,(\delta x)^2 + (\delta x)^3 \\ \delta y &= x^3 + 3x^2\,\delta x + 3x\,(\delta x)^2 + (\delta x)^3 - x^3\ (=y) \\ &= 3x^2\,\delta x + 3x\,(\delta x)^2 + (\delta x)^3 \end{aligned}$$

As δx is very small, then powers of this quantity may be neglected.

$$\therefore \quad \frac{\delta y}{\delta x} = 3x^2$$

In the limit as $\delta x \longrightarrow 0$

$$\frac{dy}{dx} = \frac{d\,(fx)}{dx} = 3x^2$$

In general if $y = x^n$, $\dfrac{dy}{dx} = nx^{n-1}$

Introducing a constant b, where b is not a function of x, then if

$$y = bx^n, \quad \frac{dy}{dx} = nbx^{n-1}$$

The term dy/dx is the first differential coefficient, or first derivative, and sometimes denoted by $f^{\text{I}}(x)$.

Successive differentiation is carried out in exactly the same manner.

Thus $y = f(x) = x^n$

$$\therefore \quad \frac{dy}{dx} = f^{\text{I}}(x) = nx^{n-1}$$

$$\frac{d^2y}{dx^2} = f^{\text{II}}(x) = n(n-1)\,x^{n-2}$$

$$\frac{d^3y}{dx^3} = f^{\text{III}}(x) = n(n-1)(n-2)\,x^{n-3}$$

$$\frac{d^4y}{dx^4} = f^{\text{IV}}(x) = n(n-1)(n-2)(n-3)\,x^{n-4}$$

$$\frac{d^5y}{dx^5} = f^{\text{V}}(x) = n(n-1)(n-2)(n-3)(n-4)\,x^{n-5}$$

etc.

Suppose $y = u^n$, where u is a function of x. This is known as a function of a function.

Then $\dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dx}$

RATES OF CHANGE.

If y is a function of time, t $(= f(t))$, then the value of y at a time $t + \delta t$ is $y + \delta y$.

During the time interval from t to $t + \delta t$, y changes at the rate of $\delta y/\delta t$.

As δt tends to the limit, $\delta t = 0$

$$\frac{\delta y}{\delta t} \text{ becomes } \frac{dy}{dt}$$

Referring to Fig. 1, let Q be a point on the line OX, moving in a direction O to X, then at a time t, the velocity of Q $= dx/dt$.

Thus $V = dx/dt$.

The acceleration is the rate of change of velocity. With respect to time $= dv/dt$

$$\therefore \quad \text{Acceleration} = \frac{dv}{dt} = \frac{d^2x}{dt^2}$$

Fig. 1.—To illustrate rate of change.

The following is a list of some important differential coefficients.

$y = e^x$ [e is the base of natural logarithms].

$$e^x = 1 + x + \frac{x^2}{\underline{|2}} + \frac{x^3}{\underline{|3}} + \frac{x^4}{\underline{|4}} - - - - \frac{x^n}{\underline{|n}}$$

$$\frac{d}{dx}(e^x) = 0 + 1 + x + \frac{x^2}{\underline{|2}} + \frac{x^3}{\underline{|3}} - - - \frac{x^n}{\underline{|n}}$$

$$\therefore \quad \frac{d}{dx}(e^x) = \frac{dy}{dx} = e^x$$

$y = u\,v$ [y, v and u being functions of x].

Let each function be given a small increment.

Then $y + \Delta y = (u + \Delta u)(v + \Delta v)$

$$y + \Delta y = uv + u\Delta v + v\Delta u + \Delta u\,\Delta v$$

$$y = uv$$

$$\therefore \quad \frac{\Delta y}{\Delta x} = \frac{u\Delta v}{\Delta x} + \frac{v\Delta u}{\Delta x} + \frac{\Delta u\,\Delta v}{\Delta x}$$

$$\frac{dy}{dx} = \frac{\Delta y}{\Delta x} \text{ as } \Delta x \longrightarrow 0$$

$$\therefore \quad \frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$$

$y = \frac{u}{v}$ [obeying the same premise as before].

$$y + \Delta y = \frac{u + \Delta u}{v + \Delta v}$$

$$\Delta y = \frac{u + \Delta u}{v + \Delta v} - \frac{u}{v}$$

$$= \frac{uv + v\Delta u - uv - u\Delta v}{v(v + \Delta v)}$$

$$= \frac{v\Delta u - u\Delta v}{v(v + \Delta v)}$$

$$\frac{\Delta y}{\Delta x} = \frac{v\,\dfrac{\Delta u}{\Delta x} - u\,.\,\dfrac{\Delta v}{\Delta x}}{v\,(v + \Delta v)}$$

$$\frac{dy}{dx} = \frac{\Delta y}{\Delta x} \text{ as } \Delta x \longrightarrow 0$$

$$\therefore \frac{dy}{dx} = \frac{v\,\dfrac{du}{dx} - u\,\dfrac{dv}{dx}}{V^2}$$

PARTIAL DIFFERENTIAL.

It is found in various applications of mathematics, that functions of more than one variable occur frequently.

Suppose $u = f\,(x\,.\,y)$

In this type of differentiation a different differential coefficient symbol is used; it is a "curly" d, thus :—∂.

To find $f^1\,(x\,y)$, each variable is differentiated separately.

Let $u = a\,x^n + b\,x^m y + C\,x\,y^p$

$$\frac{\partial u}{\partial x} = n\,a\,x^{n-1} + m\,b\,x^{m-1}y + Cy^p$$

$$\frac{\partial u}{\partial y} = b\,x^m + p\,C\,x\,y^{p-1}$$

INTEGRATION.

Integration is the reverse of differentiation. Let $f\,(x)$ be a function of x.

$$f\,(x) = \int (x) = \int f\,(x)\;dx$$

$f\,(x)$ is the integrand and x is the variable of integration. This type of integral is known as indefinite. The insertion of limits transforms it into a definite integral, *e.g.*,

$$\int_{x=0}^{x=a} f\,(x)\;dx$$

If b is a constant (*i.e.*, not a function of x), then

$$\int b\,f\,(x)\;dx = b\int f\,(x)\;dx$$

Example : $\int x^n \, dx = \dfrac{x^{n+1}}{n+1}$ $\Big[$provided n is not $= -1\Big]$

SUCCESSIVE INTEGRATION.

Suppose the density p of a plate varies as its dimensions x and y.

Then p may be expressed as a function of these two quantities, $p = f\,(x.y.)$

The evaluation of this quantity leads to a double integral,

$$\int_a^b \int_c^d f\,(x\,y)\ dx\ dy$$

Similarly, if the density varies also as the thickness t, then a triple integral results.

thus ; $$\int_a^b \int_c^d \int_e^f \quad f\,(x\,y\,t)\ dx\ dy\ dt$$

Successive integration may also be applied to a particular function.

In the case of beam problems, it is known that :

$$\text{Curvature} = \frac{1}{\text{E I}}\ \text{M}$$

$$\text{Slope} = \int \frac{1}{\text{E I}}\ \text{M}\ dx$$

$$\text{Deflection} = \int \int \frac{1}{\text{E I}}\ \text{M}\ dx\ dx$$

(Problems of a similar nature will be discussed in more detail in another part of this section).

INTEGRATION BY PARTS.

It has been shown that :

$$\frac{d}{dx}(u\,v) = u\,\frac{dv}{dx} + v\,\frac{du}{dx}\,.$$

$$\therefore \quad -u\,\frac{dv}{dx} = v\,\frac{du}{dx} - \frac{d}{dx}\,u\,v$$

Multiplying by -1 and integrating,

$$\int u\,\frac{dv}{dx}\,dx = u\,v - \int v\,\frac{du}{dx}\,dx$$

DIFFERENTIAL EQUATIONS.

These form a very important part in the solving of many problems arising in electricity, mechanics, chemistry, physics, etc. An equation containing derivatives of a variable is called a differential equation.

The order of a differential equation depends upon the power to which the highest derivative is raised.

The degree of a differential equation is the degree of the highest differential coefficient.

As an illustration of these definitions, consider the following equations :—

(a) $$a \frac{d^3y}{dt^3} + b \frac{d^2y}{dt^2} + c \frac{dy}{dt} = f(t).$$

This is of the third order, and first degree.

(b) $$\frac{d^2y}{dx^2} + m \frac{dy}{dx} = f(x).$$

This is of the second order, and first degree.

(c) $$\frac{dy}{dt} + t = f(t)$$

This is of the first order, and first degree.

(d) $$\left(\frac{d^2y}{dx^2}\right)^2 + \frac{dy}{dx} + y = f(x)$$

This is of the second order, and second degree.

(e) $$\left(\frac{d^4y}{dt^4}\right)^3 + \frac{d^3y}{dt^3} + \frac{d^2y}{dt^2} + \frac{dy}{dt} + C = f(t)$$

This is of the fourth order, and third degree.

All the above equations are ordinary differential equations. A partial differential equation is one involving partial differential coefficients.

$$e.g., \quad \alpha^2 \frac{\partial^2 y}{\partial m^2} + \beta^2 \frac{\partial y}{\partial m} = a \frac{\partial^2 y}{\partial x^2} + b \frac{\partial y}{\partial x}$$

Types of Differential Equations.

(i) EXACT DIFFERENTIAL EQUATIONS.

$$x \cdot dy + y \cdot dx = 0$$
$$\therefore d(x \cdot y) = 0$$

Integrating, $x \cdot y = C$ (where C is a constant).

Example :—

$$L\frac{di}{dt} + Ri = E$$

Multiplying by $e^{R/L\cdot t}$

$$L\frac{di}{dt}\,e^{R/L\cdot t} + Ri\,.\,e^{R/L\cdot t} = E\,.\,e^{R/L\cdot}$$

$$\frac{di}{dt}\,e^{R/L\cdot t} + \frac{R}{L}\,i\,.\,e^{R/L\cdot t} = \frac{E}{L}\,.\,e^{R/L\cdot}$$

$$\therefore \quad \frac{d}{dt}\,.\,i\,.\,e^{R/L\cdot} = \frac{E}{L}\,.\,e^{R/L\cdot t}$$

Integrating, $$i\,.\,e^{R/L\cdot t} = \frac{E}{L}\,.\,\frac{L}{R}\,.\,e^{-R/L\cdot t} + K$$

Multiplying by $e^{R/L\cdot t}$

$$i = \frac{E}{R} + K\,.\,e^{-R/L\cdot t}$$

(*ii*) SOLUTION BY SEPARATION OF VARIABLES.

$$L\frac{di}{dt} + Ri = E$$

$$\frac{di}{dt} = \frac{E - Ri}{L}$$

$$\therefore \quad \frac{1}{-E + Ri}\,.\,di = -\frac{1}{L}\,.\,dt$$

$$\frac{1}{-E/R + i}\,.\,di = -\frac{R}{L}\,.\,dt$$

Integrating, $$\log.\left[-\frac{E}{R} + i\right] = -\frac{R}{L}\,.\,t + A$$

$$\therefore \quad -\frac{E}{R} + i = e^{A}\,e^{R/L\cdot t}$$

let $e^{A} = K$

$$\therefore \quad i = \frac{E}{R} + K\,e^{-R/L\cdot t}$$

(*iii*) Homogeneous Equations.

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right)$$

$$\text{let} \quad v = \frac{y}{x} \qquad \therefore \; y = vx$$

$$\frac{dy}{dx} = v + x \,.\, \frac{dv}{dx}$$

$$v + x \,.\, \frac{dv}{dx} = f(v)$$

$$x \,.\, \frac{dv}{dx} = f(v) - v$$

$$\frac{1}{f(v) - v} \,.\, dv = \frac{1}{x} \,.\, dx$$

Integrating

$$\int \frac{1}{f(v) - v} \,.\, dv = \int \frac{1}{x} \,.\, dx = \log x + C$$

Example.

$$\frac{dy}{dx} = \frac{x^2 + y^2}{2\,xy}$$

$$v = \frac{y}{x}$$

$$\frac{dy}{dx} = v + x\frac{dv}{dx} = \frac{1 + v^2}{2\,v}$$

$$x \,.\, \frac{dv}{dx} = \frac{1 + v^2}{2\,v} - v = \frac{1 - v^2}{2\,v}$$

$$\therefore \quad \frac{2\,v}{1 - v^2} \,.\, dv = \frac{1}{x} \,.\, dx$$

$$-\log(1 - v^2) = \log x + \log C$$

$$\therefore \quad (1 - v^2) + x + C = 0$$

$$\therefore \quad \frac{x^2 - y^2 + x^3}{x^2} + C = 0$$

(*iv*) LINEAR EQUATIONS.

$$\frac{dy}{dx} + Py = Q$$

Reducing this equation by equating Q to zero, we have :

$$\frac{dy}{dx} + Py = 0$$

$$\frac{dy}{y} = -P \, . \, dx$$

Integrating, $\log . y = -\int P \, . \, dx + \log . C$

$\because$ $y = e^{-\int P \, . \, dx} C$

$\therefore$ $C = y \, e^{\int P \, . \, dx}$

Differentiating,

$$e^{\int P . dx} \left[\frac{dy}{dx} + Py\right] = 0$$

also, $$e^{\int P . dx} \left[\frac{dy}{dx} + Py\right] = Q \, e^{\int P . dx}$$

$$\frac{d}{dx} \, . \, (y \, . \, e^{\int P . dx}) = Q \, e^{\int P . dx}$$

Integrating,

$$y \, e^{\int P . dx} = \int Q \, e^{\int P . dx} \, dx + K$$

$$y = e^{-\int P . dx} \left[\int Q \, e^{\int P . dx} \, dx + K\right]$$

Example.

Consider the equation, L di/dt + Ri = E, the solution of which has been found previously :

$$i = e^{-R/L . t} \left[\int \frac{E}{L} \, . \, e^{R/L . t} \, . \, dt + K\right]$$

E is constant (D.C. case).

If at $t = 0$, $i = 0$,
then :

$$i = \frac{E}{R}\left[1 - e^{-R/L . t}\right]$$

(v) LINEAR DIFFERENTIAL EQUATION HAVING CONSTANT COEFFICIENTS.

$$a \cdot \frac{d^2y}{dx^2} + b \cdot \frac{dy}{dx} + cy = 0$$

The following may be noted in connection with the above equation :

(a) The coefficients a, b and c are constant.
(b) The equation is of the second order.
(c) The equation is linear, *i.e.*, the highest derivative is of the first degree.

Example.

$$\frac{d^2y}{dt^2} + 7 \cdot \frac{dy}{dt} + 10y = 0$$

Let y be of the form e^{mt}

Thus, $\frac{dy}{dt} = m \cdot e^{mt}$, $\frac{d^2y}{dt^2} = m^2 \cdot e^{mt}$

The equation may now be written :

$$m^2 \cdot e^{mt} + 7m \cdot e^{mt} + 10 \cdot e^{mt} = 0$$

Dividing by e^{mt} $[e^{mt} \neq 0]$

$$m^2 + 7m + 10 = 0$$

$\therefore \quad m = -5$ or -2

$\therefore \quad y = Xe^{-5t} + Ye^{-2t}$

To Check.

$$\frac{dy}{dt} = -5\,Xe^{-5t} - 2\,Ye^{-2t}$$

$$\frac{d^2y}{dt^2} = +25\,Xe^{-5t} + 4\,Ye^{-2t}$$

$$\therefore \quad \frac{d^2y}{dt^2} + 7\frac{dy}{dt} + 10y = 25\,Xe^{-5t} + 4\,Ye^{-2t} - 35\,Xe^{-5t}$$
$$- 14\,Ye^{-2t} + 10\,Xe^{-5t} + 10\,Ye^{-2t}$$
$$= 0$$

Consider an equation of the second order,

$$\frac{d^2y}{dt^2} + n^2y = 0$$

This may be written :

$$\begin{aligned} m^2 + n^2 &= 0 \\ m^2 &= -n^2 \\ m &= \pm j n \quad [j = \sqrt{-1}] \\ \therefore \quad y &= A e^{jnt} + B e^{-jnt} \\ &= A[\cos nt + j \sin nt] + B[\cos nt - j \sin nt] \\ &= (A+B) \cos nt + j(A-B) \sin nt \\ &= C \cos nt + D \sin nt \end{aligned}$$

Application to a Mechanical Pendulum.

From Newton's Second Law of Motion, $F = d(mv)/dt$

$$= m \cdot \frac{dv}{dt} + v \cdot \frac{dm}{dt}$$

If the mass is constant, the second member of the right-hand equation is zero.

Force = mass × acceleration.

It is usual in examples of this kind to use a "dot" notation for differentiation.

$\therefore$ if x = displacement

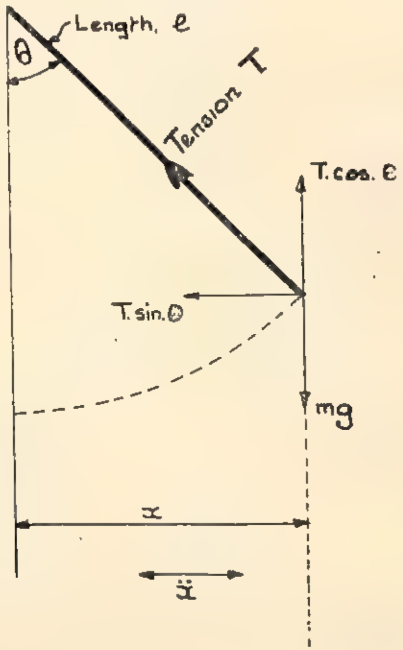


Fig. 2.—Mechanical Pendulum (Newton's second law)

$$\text{velocity} \quad v = \frac{dx}{dt} = \dot{x}$$

$$\text{acceleration} \quad = \frac{d^2x}{dt^2} = \ddot{x}$$

Referring to Fig. 2 :

$$\begin{aligned} T \cos \theta &= m g \\ - T \sin \theta &= m \ddot{x} \\ \text{length of arc} &= l\,\theta \end{aligned}$$

$$\begin{aligned} \text{Thus the Restoring Force} &= m\,l\,\ddot{\theta} \\ x = l \sin \theta &= l\,\theta \ [\theta \text{ is assumed small}] \\ \ddot{x} &= l\,\ddot{\theta} \\ \tan \theta &= \ddot{\theta}\,(-l/g) \\ \ddot{\theta} + g/l \tan \theta &= 0 \\ \therefore \quad \ddot{\theta} + g/l\ \theta &= 0 \end{aligned}$$

This may be compared with :

$$\frac{d^2y}{d\,t^2} + n^2y = 0$$

The solution is, with the premise,

$$\theta = \alpha, \quad \dot{\theta} = 0 \text{ when } t = 0$$
$$\theta = \alpha \cos t\,\sqrt{g/l}$$

Alternative method, involving energy considerations, where θ is not assumed small.

Fig. 3 is an elaboration of Fig. 2.

It is known that potential energy + kinetic energy = constant.

$$\begin{aligned} \text{P.E.} &= g\,m\,l\,(1 - \cos \theta) \\ \text{K.E.} &= \tfrac{1}{2}\,m\,(l\,\dot{\theta})^2 \end{aligned}$$

$$\therefore \quad g\,m\,l\,(1 - \cos \theta) + \tfrac{1}{2}\,m\,l^2\,\dot{\theta}^2 = K$$

Differentiate to eliminate K.

$$= g\,m\,l \sin \theta + \tfrac{1}{2}\,m\,l^2\,2 \quad \left(\frac{d\theta}{dt}\right) \frac{d}{d\theta} \cdot \frac{d\theta}{dt} = 0$$

$$= g\,m\,l \sin \theta + m\,l^2\,\ddot{\theta} = 0$$
$$= \ddot{\theta} + g/l \sin \theta = 0 \quad \text{(as previous solution).}$$
$$m\,g\,l\,(1 - \cos \theta) + \tfrac{1}{2}\,m\,(l\,\dot{\theta})^2 = K$$
$$(\dot{\theta})^2 = 2\,n^2 \cos \theta + K \quad [n = \sqrt{g/l}]$$

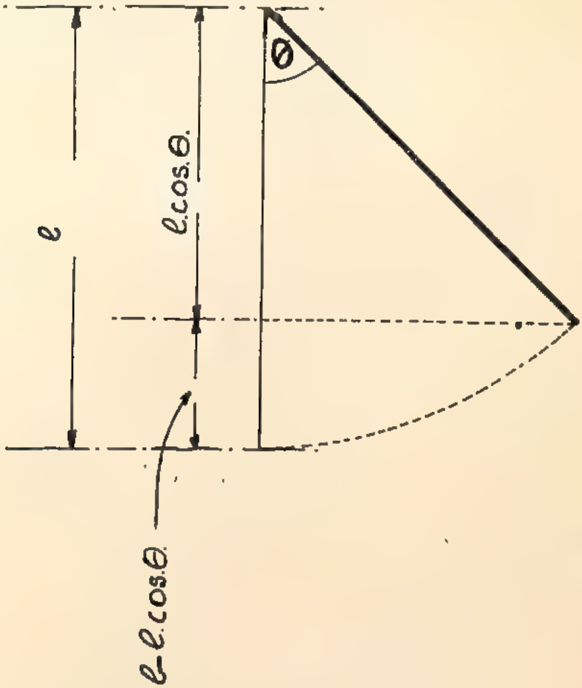


Fig. 3.—Mechanical Pendulum (Energy method).

Let α be angular extent of swing,

$$[\therefore \quad \doteqdot \hat{\theta}]$$

$$
\begin{aligned}
\text{Thus} \qquad \theta &= \alpha \text{ when } \theta = 0 \\
0 &= 2\,n^2 \cos \alpha + \mathrm{K} \\
\mathrm{K} &= -2\,n^2 \cos \alpha \\
\therefore \qquad (\theta)^2 &= 2\,n^2 \,[\cos \theta - \cos \alpha] \\
\cos \alpha &= 1 - 2 \sin^2 (\alpha/2) \\
\cos \theta &= 1 - 2 \sin^2 (\theta/2) \\
\therefore \qquad (\dot{\theta})^2 &= 4\,n^2 \,(\sin^2 (\alpha/2) - \sin^2 (\theta/2)\,) \\
\dot{\theta} &= 2\,n \sqrt{[\sin^2 (\alpha/2) - \sin^2 (\theta/2)\,]}
\end{aligned}
$$

$$\int n\,.\,dt = \int \frac{d\,\theta}{2 \sqrt{[\sin^2 (\alpha/2) - \sin^2 (\theta/2)\,]}}$$

The solution of this involves the use of elliptic integrals* and the time of a complete oscillation is given by

$$T = 4 \cdot \frac{1}{n} \cdot F\left[\sin \frac{\alpha}{2}\right]$$

F [sin $\alpha/2$] is the elliptic integral of the first kind (modulus sin $\alpha/2$).

Expressing as an infinite series :

$$T = \frac{2\pi}{n}\left[1 + \frac{1^2}{2^2}\sin^2\left(\frac{\alpha}{2}\right) + \frac{1^2}{2^2}\cdot\frac{3^2}{4^2}\cdot\sin^4\left(\frac{\alpha}{2}\right) + \frac{1^2}{2^2}\cdot\frac{3^2}{4^2}\cdot\frac{5^2}{6^2}\sin^6\left(\frac{\alpha}{2}\right) + \ldots\right]$$

A first approximation gives :

$$T = 2\pi/n = 2\pi\sqrt{l/g}$$

Comparisons between Electrical and Mechanical Systems.

Resistance.

(a) Electrical.

Effect : to cause dissipation.

R = e/i

where R = resistance (abohms).

e = voltage across resistance (abvolts).

i = current flowing through circuit (abamps).

(b) Mechanical (Friction).

Effect : to cause dissipation of energy.

P = f/v

p = mechanical resistance.

f = applied force (dynes).

v = velocity (cms./sec.).

* $\int A\,[\,a\sqrt{\phi\, a}\,]\, da$ A is a rational function. ϕ (a) is an algebraic function.

Integral of the first kind.

$$F(8, \omega) = \int_{a=0}^{a=a} \frac{d\,a}{\sqrt{(1-a^2)(1-\omega^2 a^2)}}$$

$$= \int_{a=0}^{8=8} \frac{d\,8}{\sqrt{1-\omega^2 \sin^2 8}}$$

Provided the assumption $a = \sin^{-1} 8$, and $\omega^2 < 1$, holds.

Inductance.

(a) ELECTRICAL.

Effect : to oppose any change in current.

$$e = L \cdot \frac{di}{dt}$$

e = voltage across inductance (abvolts).
L = Inductance (abhenries).
$\frac{di}{dt}$ Rate of change of i (abamps./sec.).

(b) MECHANICAL (Inertia).

Effect : to oppose a change in velocity.

$$f = m \cdot \frac{dv}{dt}$$

m = mass (grams).
f = applied force (dynes).
$\frac{dv}{dt}$ = rate of change of velocity (cms./sec./sec.).

Capacitance.

(a) ELECTRICAL.

Effect : to oppose a change in voltage.

$$i = C \cdot \frac{de}{dt}$$

C = capacitance (abfarads).
i = current (abamps.).
$\frac{de}{dt}$ = rate of change of voltage (abvolts./sec.).

$$e = \frac{1}{C} \int i \cdot dt = \frac{q}{C}$$

where q ($=\int i \cdot dt$) = charge on condenser . . . . . abscoulombs.

(b) MECHANICAL. (Stiffness Compliance).

$$f = \frac{x}{C} = S$$

C = compliance	-	-	(cms./dyne).
S = 1/C stiffness	-	-	(dynes/cms.)
f = applied force	-	-	(dynes).
x = displacement	-	-	(cms.).

DEGREES OF FREEDOM IN SYSTEMS.

In Fig. 4 a mechanical and electrical system is shown, each possessing one degree of freedom.

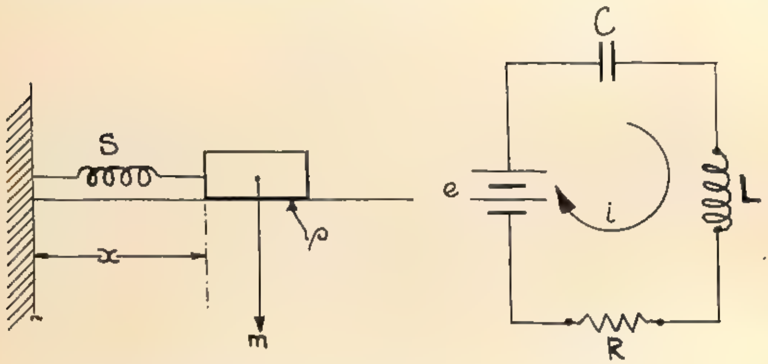


Fig. 4.—To illustrate single degree of freedom in mechanical and electrical systems.

Fig. 5 shows two similar systems, each possessing two degrees of freedom.

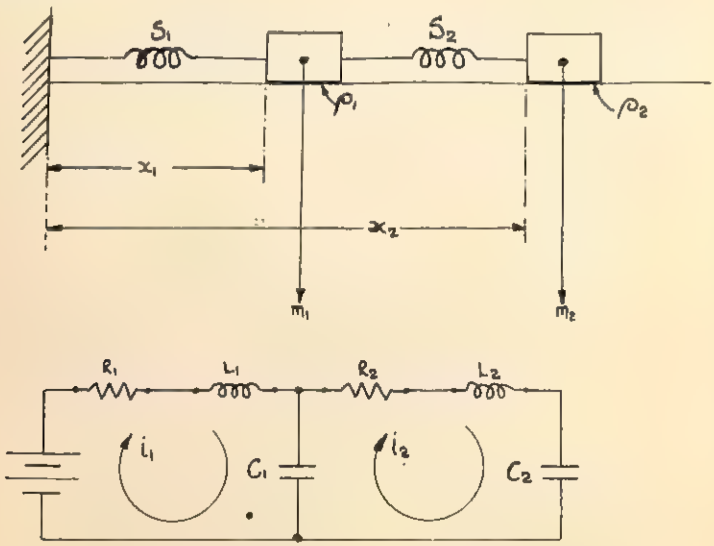


Fig. 5.—Double degrees of freedom.

Equations to be used in the Analysis of Systems.

MECHANICAL : Newton's Laws.

e.g., $m\ddot{x}$ = applied force − Restoring and frictional forces.
$I\ddot{\theta}$ = applied torque—Restoring torque and Frictional forces.

ELECTRICAL : Kirchoff's Laws.

e.g., Σ applied forces (e.m.f.'s) = Σ back (e.m.f.'s).

PARTS OF A DIFFERENTIAL EQUATION.

The equation, $am^2 + bm + c = 0$ is the *Auxiliary Equation.*

A solution of the form, $y = Ae^{nx} + Be^{-nx}$, where A and B are arbitary constants, is the *Complementary Function.*

When values are substituted for these arbitary constants, the solution is the *Particular Integral.*

BREAKDOWN OF EQUATIONS.

Consider the equation,

$$\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y = 0$$

This reduces to : $m^2 + 6m + 9 = 0$

$$\therefore \quad (m+3)^2 = 0$$

$$\therefore \quad y = Ae^{-3x} + Be^{-3x}$$
$$= [A+B]\,e^{-3x} = K\,e^{-3x}$$

Wherever the roots of the equation are equal, the equation must be reduced to the form, $y = e^{mx}\,[Ax + B]$.

Consider the equation :

$$\frac{d^2y}{dt^2} + a\,.\frac{dy}{dt} + b\,.\,y = K\,e^{\beta t}$$

If β is a root of the auxiliary equation used in obtaining the complementary function, then the particular integral cannot be found by making the substitution $y = A\,e^{\beta t}$.

The substitution, $y = At\,e^{\beta t}$ must be made.

APPLICATION OF THE CALCULUS TO BENDING PROBLEMS.

The following relationship is derived from the theory of simple bending :

$$\frac{1}{R} = \frac{M}{EI} \qquad \left[\frac{M}{I} = \frac{f}{y} = \right]$$

$$\frac{1}{R} = \frac{d^2y}{dx^2} \quad \text{(when } dy/dx \text{ is negligible).}$$

Thus, $$\frac{d^2y}{dx^2} = \frac{M}{EI} \quad \text{or } EI \,.\, \frac{d^2y}{dx^2} = M$$

Also, $$\frac{d^4y}{dx^4} = \frac{1}{EI} \,.\, \omega \qquad \text{(Loading)}$$

$$\frac{d^3y}{dx^3} = \frac{1}{EI} \int \omega \,.\, dx \qquad \text{[Shearing force (S.F.)]}$$

$$\frac{d^2y}{dx^2} = \frac{1}{EI} \iint \omega \,.\, dx \,.\, dx \quad \text{[Bending moment (M)]}$$

$$\frac{dy}{dx} = \frac{1}{EI} \iiint \omega \,.\, dx \,.\, dx \,.\, dx \qquad \text{[Gradient].}$$

$$y = \frac{1}{EI} \iiiint \omega \,.\, dx \,.\, dx \,.\, dx \,.\, dx \qquad \text{[Deflection]}$$

(1) With reference to Fig. 6,

$$M = \frac{\omega}{2} (l-x)^2$$

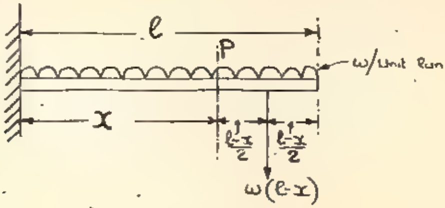
$$EI \frac{d^2y}{dx^2} = \frac{\omega}{2} (l-x)^2$$

$$EI \frac{dy}{dx} = -\frac{\omega}{6} (l-x)^3 + A$$

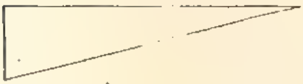
When $x = 0$ $\dfrac{dy}{dx} = 0$

$$0 = -\frac{\omega l^3}{6} + A \qquad \therefore \quad A = \frac{\omega l^3}{6}$$

$$\therefore \quad EI \frac{dy}{dx} = -\frac{\omega}{6} (l-x)^3 + \frac{\omega l^3}{6}$$



S.F.



M.

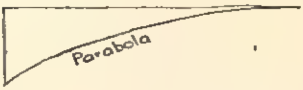


Fig. 6.—Cantilever Beam, uniform load.

$$E\,I\;y = \frac{\omega}{24}(l-x)^4 + \frac{\omega\, l^3 x}{6} + B$$

when $x=0$, $y=0$.

$$\therefore\; 0 = \frac{\omega\, l^4}{24} + B \qquad \therefore\; B = -\frac{\omega\, l^4}{24}$$

$$E\,I\;y = \frac{\omega}{24}(l-x)^4 + \frac{\omega\, l^3 x}{6} - \frac{\omega\, l^4}{24}$$

$$E\,I\;y\;(\text{at } x=l) = \frac{\omega\, l^4}{6} - \frac{\omega\, l^4}{24}$$

$$= \frac{\omega\, l^4}{8}$$

$$\omega l = W$$

$$\therefore\; y = \frac{W\, l^3}{8\,.\,E\,I}$$

(2) With reference to Fig. 7,

$$M = EI\frac{d^2y}{dx^2} = \frac{\omega}{2}(l-x)^2 - \frac{l\omega}{2}(l-x).$$

$$EI\frac{dy}{dx} = -\ \omega\ (l-x)^3 + \frac{l\omega}{4}(l-x)^2 + A$$

$$EI\,y = \frac{\omega}{24}(l-x)^4 - \frac{l\omega}{12}(l-x)^3 + Ax + B$$

$$\left.\begin{aligned} \text{when } x &= 0, \quad y = 0 \\ x &= l, \quad y = 0 \end{aligned}\right\}$$

$$EI\,y \text{ (at } x = 0) = \frac{\omega l^4}{24} - \frac{\omega l^4}{12} + B = 0$$

$$EI\,y \text{ (at } x = l) = A \cdot l + B = 0$$

$$\therefore\ B = \frac{\omega l^4}{24} \text{ and } A = -\frac{\omega l^3}{24}$$

$$EI\,y = \frac{\omega}{24}(l-x)^4 - \frac{\omega l\,(l-x)^3}{12} - \frac{\omega l^3 x}{24} + \frac{\omega l^4}{24}$$

$$EI\,y \text{ (at } x = \frac{l}{2}) = \frac{\omega l^4}{384} - \frac{\omega l^4}{96} - \frac{\omega l^4}{48} + \frac{\omega l^4}{24}$$

$$\omega l = W$$

$$\therefore\ y = \frac{5}{384} : \frac{Wl^3}{EI}$$

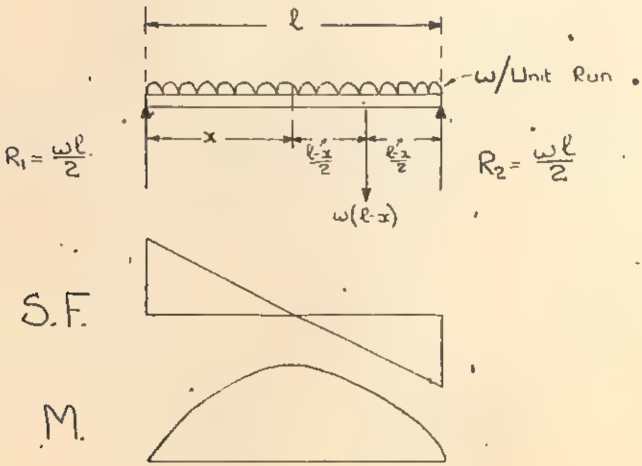


Fig. 7.—Simple Beam, uniform load.

(3) Reference to Fig. 8.

$$M \text{ (at P)} = \frac{W}{2} \cdot x$$

$$E\,I \frac{d^2y}{dx^2} = \frac{W}{2} \cdot x$$

$$y \text{ (at } x = l/2) = \hat{y} = \frac{W\,l^3}{48\,E\,I}$$

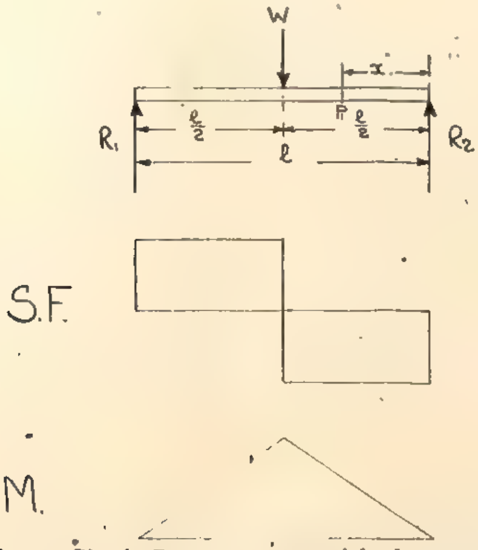


Fig. 8.—Simple Beam, concentrated load at centre.

PARTIAL DIFFERENTIAL EQUATIONS.

In problems pertaining to wave-motion, the following equations exist :—

$y = f\,(x - vt)$ representing motion to the right.

$y = F\,(x + vt)$ representing motion to the left.

The equation $y = f\,(x - vt) + F\,(x + vt)$ represents the double wave motion.

$$\frac{\partial y}{\partial x}\, f'\,(x - vt) + F'\,(x + vt)$$

$$\frac{\partial y}{\partial t} = -v f'\,(x - vt) + v\,F'\,(x + vt)$$

$$\frac{\partial^2 y}{\partial x^2} = f''(x-vt) + F''(x+vt)$$

$$\frac{\partial^2 y}{\partial t^2} = v^2 f''(x-vt) + v^2 F''(x+vt)$$

$$= v^2 \left[\frac{\partial^2 y}{\partial x^2}\right].$$

$$\therefore \quad \frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2} \left[\frac{\partial^2 y}{\partial t^2}\right] \tag{1}$$

Application of above Equation.

Considering a section of an "ideal" transmission line having a capacitance of C. farads per unit length, and an inductance L. henries per unit length. On a section of the line, δx, in length,

$$\text{Current} = C \frac{\partial e}{\partial t} = \frac{\partial i}{\partial x}.$$

$$\text{Voltage} = \frac{\partial e}{\partial x} = L \frac{\partial i}{\partial t}.$$

Differentiating, we have :

$$\frac{\partial^2 i}{\partial x\, \partial t} = C \frac{\partial^2 e}{\partial t^2} \quad \text{[with respect to } t]$$

$$\frac{\partial^2 e}{\partial x^2} = L \frac{\partial^2 i}{\partial t\, \partial x} \quad \text{[with respect to } x]$$

Thus, $$\frac{\partial^2 e}{\partial x^2} = L C \frac{\partial^2 e}{\partial t^2} \tag{2}$$

By comparing equations (1) and (2) we have :

$$\frac{1}{v^2} = L C, \quad \therefore \quad V = \frac{1}{\sqrt{LC}}$$

The capacitance of two parallel feeders (air dielectric assumed) is :

$$C = \frac{9 \times 10^{-11}}{4 \text{ logh. } d/r} \text{ farads/cm.}$$

(where d = spacing in cms. and r = radius in cms.)

Employing the same symbols :

Inductance, $L = \text{logh } d/r \times 10^{-9}$ henries/cm.

$$V = \frac{1}{\sqrt{LC}}$$

$$= \frac{1}{\sqrt{4 \text{ logh } d/r \cdot \dfrac{1}{4 \text{ logh } d/r} \cdot 10^{-9} \cdot 9 \cdot 10^{-11}}}$$

$$= \frac{1}{\sqrt{\dfrac{1}{9 \times 10^{20}}}}$$

$$= \sqrt{9 \times 10^{20}} \qquad = \mathbf{3} \times \mathbf{10^{10}} \text{ cms./sec.}$$

This is, of course, approximately equal to the velocity of light. Thus waves travelling along air-surrounded lines attain a velocity approaching 186,000 miles/sec.

INTRODUCTION TO HEAVISIDE'S OPERATIONAL CALCULUS.

Contrary to common opinion, Heaviside's operational calculus is not a new branch of mathematics. Similar methods were used by earlier mathematicians ; its originality lies in the application.

It has already been seen that the first derivative, or differential coefficient, may be of the form, dy/dt.

This could be written $d/dt \cdot y$.

Heaviside writes for this, $p \cdot y$; thus p is equivalent to d/dt, and it is an operator of differentiation.

Similarly the symbol $1/p$ is the reverse of the symbol p. Therefore $1/p$ is the reverse of differentiation, namely integration.

$$\text{Thus,} \quad P = \frac{d}{d\,t}$$

$$\frac{1}{P} = \int_0^t [\quad] \cdot dt$$

Example.

To differentiate $3x^2 + 2x^3 + x^4$

$$\textit{i.e.,} \quad p\,[3x^2 + 2x^3 + x^4] = [6x + 6x^2 + 4x^3]$$
$$\text{or} \quad p \cdot 3x^2 + p \cdot 2x^3 + p : x^4 = 6x + 6x^2 + 4x^3$$

Thus, the operator p may be used algebraically.

Example.

Differentiate x^5.

$$p'(x^5) = p x^5 = 5x^4$$

Differentiating again, $p\ (5x^4) = 20x^3$

Differentiating again, $p\ (20x^3) = 60x^2$, etc.

$$\text{Thus,}\quad p^n = \frac{d^n}{dt^n}$$

Example.

Integrate 1, with respect to t.

$$\text{Thus,}\quad \frac{1}{p}\cdot 1 \quad = t\ \left(= \frac{t}{1}\right)$$

$$\text{Integrating again,}\quad = \frac{1}{p}\cdot(t) = \frac{t^2}{1\cdot 2} = \frac{t^2}{\lfloor 2}$$

$$\text{Integrating again}\quad = \frac{1}{p}\left[\frac{t^2}{1\ 2}\right] = \frac{t^3}{1\ 2\ 3} = \frac{t^3}{\lfloor 3}$$

$$\text{Integrating again}\quad = \frac{1}{p}\left[\frac{t^3}{1\ 2\ 3}\right] = \frac{t^4}{1\ 2\ 3\ 4} = \frac{t^4}{\lfloor 4}$$

$$\text{Thus,}\quad \frac{1}{p^n} = \frac{t^n}{\lfloor n}$$

A cautionary note must be stressed. As the symbol p is an operator, it only operates on terms following its insertion in an equation. Thus px^2 is an operation demanding the differentiation of x^2, but x^2p is not.

THE UNIT FUNCTION.

Suppose that a force is about to be applied suddenly to a system; before time $t=0$, the magnitude of the force is zero, but immediately subsequent to $t=0$, the force rises instantaneously to a specific value, and remains at that value. Thus with such a supposition, many complicated problems may be reduced to comparatively simple calculations. Heaviside visualised such a condition, and based his operational calculus on this. The specific value of this force or impulse, being unity and represented by the symbol (1). See Fig. 9.

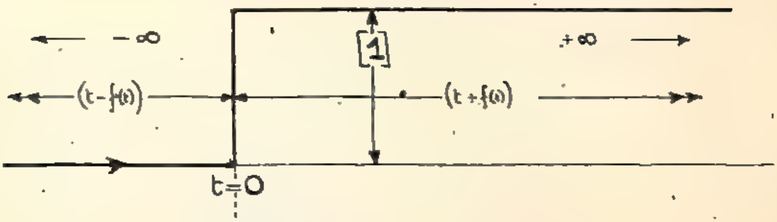


Fig. 9.—Representation of Unit function.

$[1] = 0$ at $t < 0$.
$[1[= 1$ at $t > 0$.
$[1] = [0$ to $1]$ at $t = 0$.

An example will now be demonstrated in full, by three different methods.

Example.—A "trip coil" is energised from an uni-directional current source. See Fig. 10.

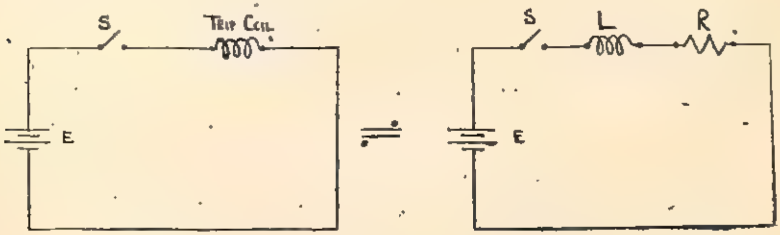


Fig. 10.—Elementary circuit of trip-coil energised from a battery.

(1) By the Ordinary Calculus.

It is known that :

$$Ri + L\,di/dt = E \tag{3}$$

Adopting the usual procedure for solving differential equations.

$$Ri + L\,di/dt = 0$$

Let $D = d/dt$

then, $(R + LD)\,i = 0$

Let i be of the form $A\,e^{mt}$

then, $di/dt = A\,m\,e^{mt}$

$\therefore$ $AR\,.\,e^{mt} + AL\,.\,m\,.\,e^{mt} = 0$

but $e^{mt} \neq 0$

$\therefore$ $AR + ALm = 0$

$ALm = -AR$

$\therefore$ $m = -R/L$

$\therefore$ $i_r = A\,e^{-R/L\,.\,t}$.

The steady-state solution of the circuit is obviously $i_s = E/R$

Therefore, the full solution is, $i_r + i_s$.

$$= \frac{E}{R} + A\, e^{-R/L\,.\,t}.$$

If the time is measured from the instant the circuit is completed, then $i=0$ when $t=0$.

$$\text{If} \qquad i = \frac{E}{R} + A\, e^{-R/L\,.\,t}.$$

$$0 = \frac{E}{R} + A\, e^{0}$$

$$\therefore \quad A = -\frac{E}{R}$$

$$\therefore \quad i = \frac{E}{R} - \frac{E}{R}\, e^{-R/L\,.\,t}.$$

$$= \frac{E}{R}\left[1 - e^{-R/L\,.\,t.}\right]$$

(2) By Heaviside's Method, leading to an Infinite Series.

$$Ri + L\, di/dt = E$$

$$[R + Lp]\, i = E \tag{4}$$

$$\therefore \quad i = \frac{E}{R + Lp}\,[1] \tag{4a}$$

$$= \frac{E}{Lp\,[\,(R/Lp) + 1\,]}\,[1] \tag{5}$$

let $a = R/L$

$$i = \frac{E}{Lp\,[\,(a/p) + 1]}\,[1]$$

$$= \frac{1}{L}\,.\,E\,.\,\frac{1}{P\,[1 + a/p]}\,[1] \tag{6}$$

Evaluating $\left[1 + \frac{a}{p}\right]^{-1}$ by the Binomial Theorem.

$$\left[1 + \frac{a}{p}\right]^{-1} = 1 - \frac{a}{p} + \frac{a^2}{p^2} - \frac{a^2}{p^3} + \frac{a^4}{p^4} \text{ etc.}$$

$$\therefore\ i = \frac{E}{L}\ \frac{1}{p}\left[1 - \frac{a}{p} + \frac{a^2}{p^2} - \frac{a^3}{p^3} + \frac{a^4}{p^4}\text{ etc.}\right][1]$$

$$= \frac{E}{L}\ \frac{1}{p}\left[1 - \frac{a\,t}{\lfloor 1} + \frac{a^2\,t^2}{\lfloor 2} - \frac{a^3\,t^3}{\lfloor 3} + \frac{a^4\,t^4}{\lfloor 4}\text{etc.}\right][1]$$

$$= \frac{E}{L}\left[t - \frac{a\,t^2}{\lfloor 2} + \frac{a^2\,t^3}{\lfloor 3} - \frac{a^3\,t^4}{\lfloor 4} + \frac{a^4\,t^5}{\lfloor 5}\text{ etc.}\right][1] \qquad (7)$$

If $a = R/L$, then $La = R$ and $L = R/a$

$$\text{then } i = \frac{Ea}{R}\left[t - \frac{a\,t^2}{\lfloor 2} + \frac{a^2\,t^3}{\lfloor 3} - \frac{a^3\,t^4}{\lfloor 4} + \frac{a^4\,t^5}{\lfloor 5}\text{ etc.}\right][1] \qquad (8)$$

$$= \frac{E}{R}\left[a\,t - \frac{a^2\,t^2}{\lfloor 2} + \frac{a^3\,t^3}{\lfloor 3} - \frac{a^4\,t^4}{\lfloor 4} + \frac{a^5\,t^5}{\lfloor 5}\text{ etc.}\right][1] \qquad (9)$$

Referring to the Infinite Series for $e^{\pm x}$ it is known that :

$$e^x = 1 + \frac{x}{\lfloor 1} + \frac{x^2}{\lfloor 2} + \frac{x^3}{\lfloor 3} + \frac{x^4}{\lfloor 4} \qquad \frac{x^n}{\lfloor n}$$

$$e^{-x} = 1 - \frac{x}{\lfloor 1} + \frac{x^2}{\lfloor 2} - \frac{x^3}{\lfloor 3} + \frac{x^4}{\lfloor 4} \qquad \pm\frac{x^n}{\lfloor n}$$

Therefore, substituting at for x, it is seen that :

$$e^{-at} = 1 - \frac{at}{\lfloor 1} + \frac{a^2t^2}{\lfloor 2} - \frac{a^3\,t^3}{\lfloor 3} + \frac{a^4\,t^4}{\lfloor 4}\text{ etc.} \qquad (10)$$

$$\therefore\ e^{-at} - 1 = -\frac{at}{\lfloor 1} + \frac{a^2\,t^2}{\lfloor 2} - \frac{a^3\,t^3}{\lfloor 3} + \frac{a^4\,t^4}{\lfloor 4}\text{ etc.} \qquad (11)$$

To obtain equation (11) in a form comparable with equation (9), multiply throughout by (-1).

$$\therefore\ -e^{-at} + 1 = \frac{at}{\lfloor 1} - \frac{a^2\,t^2}{\lfloor 2} + \frac{a^3\,t^3}{\lfloor 3} - \frac{a^4\,t^4}{\lfloor 4}\text{ etc.}$$

$$\therefore\ i = E/R\,[1 - e^{-at}]$$

$$\text{But } a = R/L$$

$$\therefore\ i = E/R\,[1 - e^{-R/L\,.\,t}]$$

The above method may seem somewhat involved, but a simplification may be made by using the Expansion Theorem.

THE EXPANSION THEOREM.

In equation (4a) the expression $i = \dfrac{E}{R + Lp}$ [1]

was deduced.

i.e., The current i = The voltage E, divided by an "impedance operator," $R + Lp$, operated on by unit function [1].

The operational solution of this may be written :

$$i = E\frac{\phi_{(p)}}{f_{(p)}} \ [1] \qquad (12)$$

where $\phi_{(p)}$ and $f_{(p)}$ are functions of p, $f_{(p)}$ may be a polynomial in p, viz. :

$$f_{(p)} = p^n + \alpha\, p^{n-a} + \beta\, p^{n-b} + \partial p^{n-c} + \delta p^{n-d} \quad x p^{n-w} \qquad (13)$$

where x is the $(w+1)$th term (obviously a, b, c, d, etc., form an A.P. series whose common difference is 1). Figures are not included, so that the series will be as general as possible).

The roots corresponding to $f_{(p)} = 0$ are assumed to be :

$$r_1, r_2, r_3 \quad - \quad - \quad - \quad r_n$$

$$\therefore\ f_{(p)} = (p - r_1)\ (p - r_2)\ (p - r_3) \qquad (p - r_n) \qquad (14)$$

$\phi_{(p)}$ may also be a polynomial in p.

If $f_{(p)}$ is of a higher power in p than $\phi_{(p)}$, then obviously $\phi_{(p)}/f_{(p)}$ will be fractional.

Conversely, if $\phi_{(p)}$ is of a higher power than $f_{(p)}$, then by division $\phi_{(p)}/f_{(p)}$ will result in a series of terms containing one fraction concerning the two functions.

From the foregoing, it is possible to write :

$$\frac{\phi_{(p)}}{f_{(p)}} = \frac{\phi_{(p)}}{(p - r_1)\ (p - r_2)\ (p - r_3)\ \ \text{etc.}} \qquad (15)$$

Solving by Partial Fractions :

$$\frac{\phi_{(p)}}{f_{(p)}} = \frac{A}{(p - r_1)} + \frac{B}{(p - r_2)} + \frac{C}{(p - r_3)} + \text{etc.} \qquad (16)$$

$$\frac{\phi_{(p)}}{f_{(p)}} = \frac{\phi_{(p)}}{(p - r_1)\, F_{(p)}} = \frac{A}{(p - r_1)} + \frac{\Phi_{(p)}}{F_{(p)}} \qquad (17)$$

Multiplying by $(p - r_1)$,

$$\frac{\phi_{(p)}}{F_{(p)}} = A + (p - r_1) \frac{\Phi_{(p)}}{F_{(p)}} \tag{18}$$

when $p = r_1$

$$\frac{\phi_{(p=r_1)}}{F_{(p=r_1)}} = A \tag{19}$$

From equation (17),

$f_{(p)} = (p - r_1)\ F_{(p)}$ [comparable with a product :

$\frac{d}{dx}(u\,v) = u\frac{dv}{dx} + v\frac{du}{dx}$]

$$\therefore\ f'_{(p)} = F_{(p)} + (p - r_1)\ F'_{(p)} \tag{20}$$

when $p = r_1$

$f'_{(p)} = F_{(p)}$

$$\therefore\ \frac{\phi_{(r_1)}}{f'_{(r_1)}} = A \tag{21}$$

Similarly :

$$\frac{\phi_{(r_2)}}{f'_{(r_2)}} = B \tag{21 a}$$

$$\frac{\phi_{(r_3)}}{f'_{(r_3)}} = C \tag{21 b}$$

and similarly for each root.

It will be appreciated that:

$$f_{(p)} = \frac{d}{dp} \cdot f_{(p)} \tag{22}$$

and that $f'_{(r_1)}$, $f'_{(r_2)}$, $f'_{(r_3)}$, etc., are the corresponding values of $f'_{(p)}$ when the respective roots are substituted.

Thus equation (12) may now be written :

$$i = E\left[\frac{A}{p - r_1} + \frac{B}{p - r_2} + \frac{C}{p - r_3} + \text{etc.}\right][1] \tag{23}$$

It has already been proved that :

$$\frac{1}{p+\alpha}[1] = \frac{1}{\alpha}\left[1 - e^{-\alpha t}\right][1]$$

Substituting $-\alpha$ for $+\alpha$

$$\frac{1}{p-\alpha}[1] = -\frac{1}{\alpha}\left[1 - e^{\alpha t}\right] [1]$$

This may be compared with equation (23).

$$\therefore\ i = E\left[-\frac{A}{r_1} - \frac{B}{r_2} - \frac{C}{r_3} - \ldots \text{ etc.}\right][1] + \left[\frac{A e^{r_1 t}}{r_1} + \frac{B e^{r_2 t}}{r_2} + \frac{C e^{r_3 t}}{r_3} + \ldots \text{etc.}\right][1] \qquad (24)$$

The first paranthesis may be compared with :

$$\left[\frac{A}{p-r_1} + \frac{B}{p-r_2} + \frac{C}{p-r_3} + \ldots\right] \text{ for the condition } p=0.$$

$$\therefore\ \left[-\frac{A}{r_1} - \frac{B}{r_2} - \frac{C}{r_3} - \ldots \text{etc.}\right] = \left[\frac{\phi_{(p)}}{f_{(p)}}\right]_{p=0} \qquad (25)$$

which may be written $\dfrac{\phi_{(0)}}{f_{(0)}}$

$\therefore\ A = \left[\dfrac{\phi_{(p)}}{f_{(p)}}\right]$ when $p = r_1$ may be written $A = \dfrac{\phi_{(r_1)}}{f_{(r_1)}}$

In the same way,

$$B = \frac{\phi_{(r_2)}}{f_{(r_2)}} \text{ etc.}$$

and $\dfrac{A}{r_1} = \left[\dfrac{\phi_{(p)}}{P f'_{(p)}}\right]$ if the condition $p = r_1$ obtains (26)

Thus it is possible to write :

$$\left\{\left[\frac{\phi_{(p)}\, e^{pt}}{p\ df/dp}\right]_{p=r_1} + \left[\frac{\phi_{(p)}\, e^{pt}}{p\ df/dp}\right]_{p=r_2} + \text{etc.}\right\}[1] \qquad (27)$$

Thus equation (24) becomes :

$$i = E\left[\frac{\phi_{(0)}}{f_{(0)}} + \sum_{r_1, r_2, r_3 \ldots} \frac{\phi_{(p)}\, e^{pt}}{p f'_{(p)}}\right][1] \qquad (28)$$

This is known as the expansion formula.

There are various methods of writing this, one of the most common being :

$$i = E \left[\frac{Y_{(0)}}{Z_{(0)}} + \sum_{p^1\, p^2\, p^3 \ldots} \frac{Y_{(p)}\, e^{pt}}{P\, dZ_{(p)}/dp}\right] [1] \qquad (29)$$

Application of the Expansion Theorem to the Example under Discussion.

$$i = E \frac{1}{R + Lp} [1] \qquad (30)$$

Re-writing in terms of equation (29), and arranging in a form conducive to application of expansion theorem,

$$i = E \frac{Y_{(p)}}{Z_{(p)}} [1] \qquad (31)$$

Thus $\quad Y_{(p)} = 1 : \quad Z_{(p)} = R + Lp$

Equating $Z_{(p)}$ to zero, one root is obtained, viz. :

$$p_1 = -\frac{R}{L}$$

$$\frac{dz}{dp} = L \text{ and } p\frac{dz}{dp} = pL$$

$$\therefore \quad \frac{Y_{(p)}}{p\ dz/dp} = \frac{1}{pL} \qquad (32)$$

when $p = p_1$

$$\left[\frac{Y_{(p)}}{p\ dz/dp}\right]_{p=p_1} = \frac{1}{-R/L\,.\,L} = -\frac{1}{R} \qquad (33)$$

and $\quad \dfrac{Y_{(0)}}{Z_{(0)}}\ (i.e.,\ p=0) = \dfrac{1}{R}$

$$\therefore \quad i = E \left[\frac{Y_{(0)}}{Z_{(0)}} + \sum_{p^1} \frac{Y_{(p)}\, e^{pt}}{p\ dz_{(p)}/dp}\right] [1] \qquad (34)$$

$$i = E \left[\frac{1}{R} - \frac{1}{R} e^{-R/L\,.\,t}\right]$$

$$i = \frac{E}{R} \left[1 - e^{-R/L\ t}\right] \qquad (35)$$

ROOTS.

In the development of the Expansion Theorem, the roots $r_1, r_2, r_3 \ldots r_n$, were extracted when $f_{(p)}$ (this may now be termed $Z_{(p)}) = 0$ The treatment accorded to these roots obviously assumed that : $r_1 \gtrless r_2 \gtrless r_3 \ldots \gtrless r_n$, but that : $r_1 \neq r_2 \neq r_3 \ldots \neq r_n \neq 0$ and that the powers of $r_1, r_2, r_3 \ldots r_n$, are integers. Depending upon the type of equation formed, many interesting variations of roots may be investigated.

EVALUATION OF A POLYNOMIAL

Due to HORNER (1819).

Supposing a value of $f(s)$ be required for specific values of s. The method described below will be found to be extremely comprehensive.

Assume a polynomial in s of degree $n=3$:

$$a s^3 + \beta s^2 + \delta s + \theta = 0$$

Arrange the coefficients as follows :—

$$\begin{array}{lllll} a & \beta & \delta & \theta & \underline{|s} \\ & \underline{a s} & \underline{a s^2 + \beta s} & \underline{a s^3 + \beta s^2 + \delta s} & \\ & a s + \beta & a s^2 + \beta s + \delta & a s^3 + \beta s^2 + \delta s + \theta & = f(s) \end{array}$$

The operation will be obvious ; namely, multiply the first coefficient (*i.e.*, the coefficient of the highest degree of the function), which is placed as a leading entry, by the particular value whose function it is required to find. The product as being placed underneath the next coefficient, and added. The result is again multiplied by s, and added to the following coefficient and so on.

A method of synthetic division may be found extremely useful for the division of, say, $f(y)$ by $(y-\alpha)$.

Let $f(y)$ be of the form : $ay^3 + by^2 + cy + d = 0$

PROCEDURE :

$$\begin{array}{lllll} a & +b & +c & +d & (\alpha) \\ \underline{\quad} & \underline{+\alpha a} & \underline{+(\alpha^3 a + xb)} & \underline{+(\alpha^3 a + \alpha^2 b + \alpha c)} & \\ a & +(\alpha a + b) & +(\alpha^2 a + \alpha b + c) & +(\alpha^3 a + \alpha^2 b + \alpha c + d) & \end{array}$$

$\therefore$ The quotient is : $ay^2 + (\alpha a + b) y + (\alpha^2 a + \alpha b + c)$
with a remainder - $\alpha^3 a + \alpha^2 b + \alpha c + d$.

Determination of Irrational Roots (Horner's Method).

$$f(a) = A a^4 + Ba^3 + Ca^2 + Da + E = 0$$

Reduce the equation by the elimination of rational roots, thus assume (see preceding method of synthetic division) the equation becomes :

$$B a^3 + D a + E = 0$$

By various methods it is possible to find the limits affecting any change in sign in $f(a)$, thus it is possible to find a root which lies between 0 and α.

Dividing $f(a)$ by $(a-\alpha)$:

$$
\begin{array}{llll|l}
B & +0 & +D & +E & (\alpha \\
 & +\alpha B & +\alpha^2 B & +\alpha^3 B+\alpha D & \\
\hline
B & +\alpha B & +\alpha^2 B+D & +\alpha^3 B+\alpha D+E & \\
 & +\alpha B & +2\alpha^2 B & & \\
\hline
B & +2\alpha B & +3\alpha^2 B+D & & \\
 & +\alpha B & & & \\
\hline
B & +3\alpha B & & &
\end{array}
$$

The transformed equation now becomes :

$$B a^3 + (3\alpha B)a^2 + (3\alpha^2 B + D)a + (\alpha^3 B + \alpha D + E) = 0,$$

and possesses a root between 0 and α. A more exact approximation of the root may now be found. By diminishing the irrational root of $F(a) = 0$ by, say, an amount δ (where δ is smaller than α), a second transformed equation may be found.

$$
\begin{array}{llll|l}
B & +3\alpha B & +(3\alpha^2 B+D) & +(\alpha^2 B+\alpha D+E) & [\delta \\
 & +\delta B & +\delta^2 B+3\delta\alpha & +\delta[(3\alpha^2 B+D)+(\delta^2 B+3\delta\alpha)] & \\
\hline
B & +B(\delta+3\alpha) & +[(3\alpha^2 B+D)+(\delta^2 B+3\delta\alpha)] & +\delta[(3\alpha^2 B+D)+(\delta^2 B+3\delta\alpha)]+(\alpha^2 B+\alpha D+E). & \\
 & +\delta B & +\delta(2\delta B+3\alpha B) & & \\
\hline
B & +(2\delta B+3\alpha B) & +\delta(2\delta B+3\alpha B)+[(3\alpha^2 B+D)+(\delta^2 B+3\delta\alpha)] & & \\
 & +\delta B & & & \\
\hline
B & +3\delta B+3\alpha B & & &
\end{array}
$$

This equation resulting from the second transformation will be found to possess a root between much narrower limits than previously. Thus the root of the original equation is :

$$\alpha + \delta + \lambda + \psi + \phi + \text{etc.},$$

where λ, ψ, ϕ . . . etc., are roots of subsequent successive transformations.

EVALUATION OF A POLYNOMIAL BY DETERMINANTS.

Assume the following simultaneous equation :

$$\alpha_1 a + \beta_1 b = x_1 \qquad (36)$$
$$\alpha_2 a + \beta_2 b = x_2 \qquad (37)$$

where a and b are unknown.

Thus :
$$\alpha_1 \beta_2 a + \beta_1 \beta_2 b = \beta_2 x_1 \qquad (36a)$$
$$\alpha_2 \beta_1 a + \beta_1 \beta_2 b = \beta_1 x_2 \qquad (37a)$$

$$(\alpha_1 \beta_2 a - \alpha_2 \beta_1 a) = (\beta_2 x_1 - \beta_1 x_2)$$

$$\text{Thus } a = \frac{\beta_2 x_1 - \beta_1 x_2}{\alpha_1 \beta_2 - \alpha_2 \beta_1}$$

Adopting the method of Determinants :

$$a = \frac{\begin{vmatrix} x_1 \cdot \beta_1 \\ x_2 \cdot \beta_2 \end{vmatrix}}{\begin{vmatrix} \alpha_1 \cdot \beta_1 \\ \alpha_2 \cdot \beta_2 \end{vmatrix}} = \frac{(x_1 \beta_2 - x_2 \beta_1)}{(\alpha_1 \beta_2 - \alpha_2 \beta_1)}$$

The method is quite obvious. Namely, coefficients downwards from left to right, constitute the positive terms (depending of course upon their individual signs) and coefficients upwards from left to right constitute the negative terms.

GENERAL CASE FOR THIRD ORDER DETERMINANTS.

$$\triangle = \begin{vmatrix} \alpha_1 \cdot \beta_1 \cdot \delta_1 \\ \alpha_2 \cdot \beta_2 \cdot \delta_2 \\ \alpha_3 \cdot \beta_3 \cdot \delta_3 \end{vmatrix} \begin{matrix} \alpha_1 \cdot \beta_1 \\ \alpha_2 \cdot \beta_1 \\ \alpha_3 \cdot \beta_3 \end{matrix}$$

The terms α, β, etc., are written to the right of the square array of elements, merely for convenience.

Thus :

$$\triangle = [\alpha_1 \beta_2 \delta_3 + \beta_1 \delta_2 \alpha_3 + \delta_1 \alpha_2 \beta_3 - \alpha_3 \beta_2 \delta_1 - \beta_3 \delta_2 \alpha_1 - \delta_3 \alpha_2 \beta_1]$$
$$[\alpha_1 (\beta_2 \delta_3 - \beta_3 \delta_2) - \alpha_2 (\beta_1 \delta_3 - \beta_3 \delta_1) + \alpha_3 (\beta_1 \delta_2 - \beta_2 \delta_1)]$$

or,

$$\triangle = \begin{vmatrix} \alpha_1 & \beta_1 & \delta_1 \\ \alpha_2 & \beta_2 & \delta_2 \\ \alpha_3 & \beta_3 & \delta_3 \end{vmatrix} = \alpha_1 \begin{vmatrix} \beta_2 & \delta_2 \\ \beta_3 & \delta_3 \end{vmatrix} - \alpha_2 \begin{vmatrix} \beta_1 & \delta_1 \\ \beta_3 & \delta_3 \end{vmatrix} + \alpha_3 \begin{vmatrix} \beta_1 & \delta_1 \\ \beta_2 & \delta_2 \end{vmatrix}$$

The lower-order determinants are found by suppressing the column and row corresponding to the particular minor.

$$\therefore \quad \triangle = a_1 [\beta_2 \delta_3 - \beta_3 \delta_2] - a_2 [\beta_1 \delta_3 - \beta_3 \delta_1] + a_3 [\beta_1 \delta_2 - \beta_2 \delta_1]$$

TYPES OF ROOTS, BY CHANGES IN SIGN.

It can be shown (*e.g.*, Lill's graphical method) that there will not be more positive roots than changes in sign of $f(a)$; similarly for negative roots with $f(-a)$.

Example.

$$f(a) = a^{10} + a^8 - 5a^7 + 8a^3 - 9a + 1 = 0$$

In this equation there are four changes of sign, so four positive roots may be expected.

$$f(-a) = a^{10} + a^8 - 5a^7 + 8a^3 - 9a + 1 = 0$$

In this, there are two changes of sign, corresponding to two negative roots.

Actually, of course, in the example discussed, there will be ten roots. Six of these have been accounted for, and it is to be expected that the other four will be imaginary.

Let $f(k)$ be of the form:

$$a(k-\alpha)(k-\beta)(k-\delta)$$

It may be that $\alpha = \beta$ or $\beta = \delta$ or $\alpha = \delta$, etc.

Obviously $(k-\alpha)^\lambda$ is a factor of $f(k)$ if α is a λ-repeated root.

$$\therefore \quad f(k) = a(k-\alpha)^\lambda (k-\beta)^\phi (k-\delta)^\psi$$

If the equation is a n'th degree equation, then $\lambda + \phi + \psi +$ succeeding powers $= n$; and all derivatives up to and including the one preceding the λ term will cease to have any significance when $k = \lambda$.

$$\therefore \quad f(k) = (k-\alpha)^\lambda \phi(k)$$
$$f'(k) = (k-\alpha)^\lambda \phi'(k) + \phi(k) \lambda (k-\alpha)^{\lambda-1}$$
$$f''(k) = (k-\alpha)^\lambda \phi''(k) + \phi'(k) \lambda (k-\alpha)^{\lambda-1} + \phi(k) \lambda(\lambda-1)(k-\alpha)^{\lambda-2} + \phi'(k) \lambda (k-\alpha)^{\lambda-1}$$

It can be seen that the first derivative contains the factor $(k-\alpha)$, $(\lambda-1)$ times; the second derivative $(\lambda-2)$ times; and it is clear that successive differentiation would give $(\lambda-3)$ times for the third derivative, until finally the λ'th derivative would eliminate the factor. This is the procedure to be adopted when repeated, or equal roots are present.

When one or more roots are equal to zero, or some roots are repeated, it is not possible to use the expansion theorem in exactly the same manner as previously outlined.

It is necessary in examples of this nature to perform an integration.

Thus, if $i = E \cdot Y_{(p)}/Z_{(p)}$ [1]
where $Z_{(p)}$ is of the form,

$$p\ (p-r_1)\ (p-r_2)\ (p-r_3),$$

then obviously one root is zero.

The procedure would then be :

$$\text{Evaluate } i_o = E \cdot \frac{Y_{(p)}}{(p-r_1)\ (p-r_2)\ (p-r_3)} \qquad [1]$$

then perform the operation, which is demanded by $1/p$, (the missing term in the equation for i_o).

The integration $\int_0^t i_o\, dt$ must be performed.

When more than one root is equal, a "dodge" may be resorted to. Each root is imagined to be slightly different.

Thus : $[p - (r_1 - \delta_1)] \neq [p - (r_1 + \delta_1)]$

As these two approach equality, the necessary corrections are made.

QUADRATIC EQUATIONS.

The equation, $ax^2 + bx + c = 0$, is a quadratic equation, *i.e.*, a polynomial in x, of the second degree.

The procedure is as follows:

$$x^2 + \frac{b}{a}x = -\frac{c}{a}$$

Add $[b/2a]^2$ to each side.

$$x^2 + \frac{b}{a}x + \left[\frac{b}{2a}\right]^2 = -\frac{c}{a} + \left[\frac{b}{2a}\right]^2$$

$$\therefore \left[x + \frac{b}{2a}\right]^2 = \frac{b^2 - 4ac}{4a^2}$$

$$\therefore \left[x + \frac{b}{2a}\right] = \frac{\pm\sqrt{b^2 - 4ac}}{2a}$$

$$\therefore \qquad x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

where $a \neq 0$

For equal roots, $b^2 = 4ac$.

For real roots, $b^2 > 4ac$.

For imaginary roots, $b^2 < 4ac$.

The quantity $(b^2 - 4\,ac)$ is known as the discriminant of the equation.

Let $b^2 - 4\,ac = w_0$

If $w_0 > 0$, roots are real and different.
If $w_0 = 0$, roots are real and equal.
If $w_0 < 0$, roots are complex and different.
If w_0 is a perfect square, $s^2 > 0$

then $x = \dfrac{-(b \pm s)}{2\,a}$ and both solutions are rational.

Whenever complex roots of a real equation occur, they occur in pairs, *e.g.*, $a \pm i\,b$. These are conjugate complex numbers. Obviously the number of complex roots must always be even.

The following problem will be discussed at length, as several important principles are involved.

Problem.—A resistance condenser and inductance are connected in series across a steady e.m.f., e.

The voltage equation is :—

$$Ri + L\,\frac{di}{dt} + \frac{1}{C}\int i\,.\,dt = E \tag{38}$$

Re-arranging and differentiating :

$$\frac{R}{L}\,.\,\frac{di}{dt} + \frac{d^2 i}{d\,t^2} + \frac{i}{L\,C} = 0 \tag{39}$$

Adopting the symbol $D = \dfrac{d}{dt}$

then :

$$\left[D^2 + \frac{R}{L}\,.\,D + \frac{1}{L\,C}\right] i = 0 \tag{40}$$

$$D = \frac{-\,R/L \pm \sqrt{R^2/L^2 - 4/LC}}{2} \tag{41}$$

$$= -\frac{R}{2\,L} \pm \sqrt{\frac{R^2}{4\,L^2} - \frac{1}{L\,C}} \tag{41a}$$

$$\therefore \quad i = Ae^{\left(-\frac{R}{2L} + \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}\right)t} + B\,e^{\left(-\frac{R}{2L} - \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}\right)t} \tag{42}$$

Let $$p_1 = \left[-\frac{R}{2L} + \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}} \right] \qquad (42a)$$

$$p_2 = \left[-\frac{R}{2L} - \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}} \right] \qquad (42b)$$

$$\therefore \quad i = A e^{p_1 t} + B e^{p_2 t} \qquad (43)$$

the term $\sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}$ may be of three forms :

(1) $\frac{R^2}{4L^2} > \frac{1}{LC}$

(2) $\frac{R^2}{4L^2} < \frac{1}{LC}$

(3) $\frac{R^2}{4L^2} = \frac{1}{LC}$

Case (1) $\frac{R^2}{4L^2} > \frac{1}{LC}$

From equation (38), if at the instant of switching-in, the condenser is not charged and the current is zero, then :

$$L \frac{di}{dt} = E$$

$$\therefore \quad \frac{di}{dt} = \frac{E}{L}$$

Thus equation (42) becomes,

$$A e^0 + B e^0 = 0 = A + B$$

It is possible to write,

$$\frac{di}{dt} = p_1 A e^{p_1 t} + p_2 B e^{p_2 t}$$

If $t = 0$

$$\frac{di}{dt} = p_1 A + p_2 B$$

But $$\frac{di}{dt} = \frac{E}{L}$$

$$\therefore \quad p_1 A + p_2 B = E/L$$

$$B = -A$$

$$\therefore \quad p_1 A - p_2 A = E/L$$

$$A(p_1 - p_2) = E/L$$

$$\therefore \quad A = \frac{E}{L} \frac{1}{(p_1 - p_2)}$$

$$\text{and} \quad B = -\frac{E}{L} \frac{1}{(p_1 - p_2)}$$

Equation (43) may be now written :

$$i = \frac{E}{L} \cdot \frac{1}{(p_1 - p_2)} e^{p_1 t} - \frac{E}{L} \cdot \frac{1}{(p_1 - p_2)} e^{p_2 t}$$

$$= \frac{E}{L} \cdot \frac{1}{(p_1 - p_2)} [e^{p_1 t} - e^{p_2 t}]$$

$$\text{But} \quad p_1 = -\frac{R}{2L} + \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}$$

$$p_2 = -\frac{R}{2L} - \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}$$

$$\therefore \quad i = \frac{E e^{-Rt/2L}}{L\, 2\sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}} \left[e^{\left(\sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}\right)t} - e^{\left(-\sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}\right)t} \right]$$

It is known that, $\sinh \phi = \dfrac{e\phi - r^{-\phi}}{2}$

$$\text{Thus,} \quad i = \frac{E\, e^{-Rt/2L}}{L\sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}} \left[\text{Sinh} \left(\sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}} \right) t \right]$$

This is an overdamped condition.

Case (1)—By Heaviside's Operational Calculus.

Equation (38) is now written,

$$Ri + Lpi + \frac{1}{p} \cdot \frac{1}{C} \cdot i = E \quad [1]$$

$$i = \frac{E}{R + Lp + \frac{1}{p} \frac{1}{C}} \quad [1]$$

$$= E \frac{C p}{R C p + L C p^2 + 1} \quad [1]$$

Thus :

$$\left.\begin{aligned} Y_{(p)} &= C p \\ Z_{(p)} &= R C p + L C p^2 + 1 \\ Y_{(o)} &= 0 \\ Z_{(o)} &= 1 \end{aligned}\right\}$$

To obtain roots of $Z_{(p)} = 0$

$$Z_{(p)} = L C p^2 + R C p + 1 = 0$$

$$\therefore \quad p = -\frac{R}{2L} \pm \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}$$

Thus :

$$p_1 = -\frac{R}{2L} + \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}$$

$$p_2 = -\frac{R}{2L} - \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}$$

$$\begin{aligned} Z'_{(p)} &= [2 L C p + R C]_{p_1, p_2} \\ &= \pm \sqrt{R^2 C^2 - 4 L C} \\ &= \pm 2 L C \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}} \end{aligned}$$

The expansion theorem from equation (29),

$$= E \left[\frac{Y_{(0)}}{Z_{(0)}} + \sum_{p_1, p_2, p_3} \frac{Y_{(p)}\, \epsilon^{(pt)}}{p \,.\, dZ_{(p)}/dp} \right] \quad [1]$$

$$= E \left[\frac{0}{1} + \sum_{p_1, p_2} \frac{C \,.\, p \,.\, e^{pt}}{p \pm 2 LC \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}} \right]$$

$$= EC\left[0 + \frac{P^1\ e^{p_1 t}}{p_1 2LC\sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}} - \frac{P^2\ e^{p_2 t}}{p_2 2LC\sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}}\right]$$

$$\text{Let } \beta = \sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}$$

$$\therefore\ i = EC\left[0 + \frac{P^1\ e^{p_1 t}}{p_1 2LC\beta} - \frac{P^2\ e^{p_2 t}}{p_2 2LC\beta}\right]$$

$$= \frac{EC}{2LC\beta}\left[\frac{p_1 e^{p_1 t}}{p_1} - \frac{p_2 e^{p_2 t}}{p_2}\right]$$

$$= \frac{E}{2L\beta}\left[e^{p_1 t} - e^{p_2 t}\right]$$

$$\frac{E\,.\,e^{-R/2L\,.\,t}}{2L\sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}}\left[e^{\left[\sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}\right]t} - e^{\left[-\sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}\right]t}\right]$$

From the previous definition of sinh. ϕ, the above equation may be reduced to :

$$i = \frac{E\,.\,e^{-R/2L\,.\,t}}{L\sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}}\left[\sinh\left(\sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}\right)t\right]$$

Case (2). $\dfrac{R^2}{4L^2} < \dfrac{1}{LC}$

In this case, $1/LC$ is greater than $R^2/4L^2$, thus the quantity inside the square root sign is negative, therefore the roots p_1 and p_2 are imaginary.

$$\therefore\ p_1 = -\frac{R}{2L} + j\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}$$

$$\text{and } p_2 = -\frac{R}{2L} - j\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}$$

(In this particular example the symbol $j = \sqrt{-1}$ is used in place of $i = \sqrt{-1}$, as ambiguity may result, by reason of the current symbol i).

Substituting in the equation,

$$i = A e^{p_1 t} + B e^{p_2 t}$$

we have

$$i = A e^{\left[-\frac{R}{2L} + j\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}\right]t} + B e^{\left[-\frac{R}{2L} - j\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}\right]t}$$

$$= e^{-R/2L.t}\left[A e^{\left[j\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}\right]t} + B e^{\left[-j\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}\right]t}\right]$$

$$\cos \theta = \frac{e^{j\theta} + e^{-j\theta}}{2}$$

similarly, $\sin \theta = \dfrac{e^{j\theta} - e^{-j\theta}}{2j}$

$$\frac{e^{j\theta} + e^{-j\theta}}{2} + j\left[\frac{e^{j\theta} - e^{-j\theta}}{2j}\right] = \frac{e^{j\theta} + e^{-j\theta}}{2} + \frac{e^{j\theta} - e^{-j\theta}}{2}$$

$$= e^{j\theta}$$

$\therefore \quad e^{j\theta} = \cos \theta + j \sin \theta$

also,

$$e^{-j\theta} = \cos \theta - j \sin \theta$$

Therefore, the expression reduces to :

$$i = e^{-R/2L.t}\left[A\left(\cos\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}\,t + j\sin\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}\,t\right)\right.$$

$$\left. + B\left(\cos\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}\,t - j\sin\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}\,t\right)\right]$$

$$= e^{-R/2L.t}\left[(A+B)\cos\sqrt{\frac{1}{LC} - \frac{R^2}{4L}}\,.\,t + j(A-B)\sin\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}\,t\right]$$

Let $A + B = C$ and $j(A - B) = D$,

then, $i = e^{-R/2L.t}\sqrt{C^2 - D^2}\sin\left(\sqrt{\dfrac{1}{LC} - \dfrac{R^2}{4L^2}}\,.\,t + \phi\right)$

Let $M = \sqrt{C^2 - D^2}$

$$\therefore \quad i = M e^{-R/2L.t}\sin\left(\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}\,.\,t + \phi\right)$$

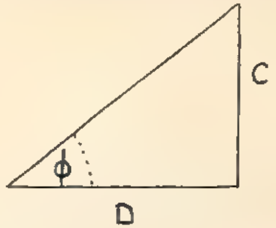


Fig. 11.—Pertaining to R.L.C. circuit (case 2).

With reference to Fig. 11,

$$\tan\phi = C/D$$
$$\text{or,}\quad \phi = \text{arc}\tan C/D.$$

With conditions obtaining as in the previous case,

$$M e^{\circ} \sin\phi = 0 \qquad (M \neq 0)$$
$$\therefore \quad \phi = 0$$

Thus $i = M e^{-R/2L \cdot t} \sin\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}\, t$

$$\therefore \quad \frac{di}{dt} = \left[M e^{-R/2L \cdot t} \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}} \cos\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}} \cdot t \right.$$
$$\left. - \frac{R}{2L} \sin\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}} \cdot t \right]$$

If $t=0$, then $\sin 0=0$, and $\cos 0=1$.

$$\frac{di}{dt} = \left[M \cdot \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}} \cdot 1 - 0 \right]$$

But $\frac{di}{dt} = \frac{E}{L}$. (see Case (1).

$$\therefore \quad M \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}} = \frac{E}{L}$$

$$\therefore \quad M = \frac{E}{L\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}}$$

The full expression now becomes :

$$i = \frac{E}{L\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}} \cdot e^{-Rt/2L} \sin \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}} \cdot t$$

This is an underdamped condition.

Case (2). By HEAVISIDE'S OPERATIONAL CALCULUS.

$$R i + L p i + 1/P \cdot 1/C \cdot i = E \quad [1]$$

$$\therefore \quad i = \frac{E\,1}{R + pL + 1/P \cdot 1/C.} \quad [1]$$

Thus $\quad Y_{(p)} = 1$

$\quad Z_{(p)} = R + pL + 1/P \cdot 1/C.$

If $\quad Z_{(p)} = 0,$

then,

$$p_1 = -R/2L + j\sqrt{1/LC - R^2/4L^2}$$

$$p_2 = -R/2L - j\sqrt{1/LC - R^2/4L^2}$$

$$Z'_{(p)} = L - 1/p^2 C$$

$$\therefore\ p Z'_{(p)} = pL - 1/pC$$

$$= \frac{p^2 LC - 1}{pC}$$

$$\Big[p\, Z'_{(p)}\Big]_{p_1} = 2Lj\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}$$

$$\Big[p\, Z'_{(p)}\Big]_{p_2} = -2Lj\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}$$

$$Y_{(o)} = 0$$

$$\therefore \quad \frac{Y_{(o)}}{Z_{(o)}} = 0$$

Substituting in the expansion formula :

$$i = E\left[0 + \frac{e^{\left[-\frac{R}{2L} + j\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}\right]t}}{2Lj\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}} - \frac{e^{\left[-\frac{R}{2L} - j\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}\right]t}}{2Lj\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}}\right]$$

$$= \frac{E\, e^{-R/2L \cdot t}}{L\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}}\left[\frac{e^{\left(j\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}\right)t} - e^{\left(-j\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}\right)t}}{2j}\right]$$

Sin θ has already been defined as :

$$\frac{e^{j\theta} - e^{j\theta}}{2j}$$

$$\therefore \; i = \frac{E\, e^{-R/2L \cdot t}}{L\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}} \left[\sin \left(\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}} \right) t \right]$$

Case (3).

$$\frac{R^2}{4L^2} = \frac{1}{LC}$$

If $\frac{R^2}{4L^2} = \frac{1}{LC}$ the expression $\sqrt{\frac{R^2}{4L^2} - \frac{1}{LC}}$

is equal to zero, therefore the roots are equal to $-R/2L$.

$$\therefore \; i = A\, e^{-R/2L \cdot t} + B\, e^{-R/2L \cdot t}$$

$$= (A + B)\, e^{-R/2L \cdot t}$$

$$\text{Thus} \quad \left(D^2 + \frac{R}{L} D \cdot + \frac{1}{LC}\right) i = 0$$

$$\therefore \left[\left(D + \frac{R}{2L}\right)^2 - \frac{R^2}{4L^2} - \frac{1}{LC} \right] i = 0$$

$$\text{But if} \quad \frac{R^2}{4L^2} = \frac{1}{LC}$$

$$\left(D + \frac{R}{2L}\right) i = 0$$

$$\text{Let } \lambda = \left[D + \frac{R}{2L} \right] i$$

$$\text{then } 0 = \lambda \left[D + \frac{R}{2L} \right]$$

From preceding discussions, $\lambda = A\, e^{-R/2L \cdot t}$

$$\text{Thus} \quad (D + R/2L)\, i = A\, e^{-R/2L \cdot t}$$

$$\therefore \; (D + R/2L)\, i\, e^{R/2L \cdot t} = A$$

as before, $D = d/dt$

$$\therefore \; i\, e^{R/2L \cdot t} = A t + C$$

where C is an integration constant.

Multiplying by $e^{-R/2L.t}$

$$i = (At + C)\,e^{-R/2L.t}$$

The boundary conditions are the same as in the preceding cases.

$$\therefore \quad E/L = A \quad \text{and} \quad C = 0$$

Thus the final expression is

$$i = E/L\,.\,t\,.\,e^{-R/2L.t}$$

This is a critically damped condition.

Case (3). By Heaviside's Operational Calculus.

$$\text{As} \quad \sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}} \longrightarrow 0$$

$$\text{then} \quad \sin\left(\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}\right)t \longrightarrow \left(\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}\right)t$$

$$i = \frac{E}{R + Lp + \frac{1}{p}\cdot\frac{1}{C}} \qquad [1]$$

Rearranging,

$$i = \frac{E}{\dfrac{RpC + Lp^2C + 1}{pC}} \qquad [1]$$

$$= \frac{pEC}{RpC + p^2LC + 1} \qquad [1]$$

$$\therefore \quad i = \frac{pEC}{CL\left[p^2 + \frac{R}{L}\cdot p + \frac{1}{LC}\right]} \qquad [1]$$

$$= \frac{pE}{L\left[p^2 + \frac{R}{L}\cdot p + \frac{1}{LC}\right]} \qquad [1]$$

$$p^2 + \frac{R}{L}\cdot p + \frac{1}{LC} = 0$$

In place of $\frac{1}{LC}$ write $\frac{R^2}{4L^2}$; this is justifiable if

$$\frac{R^2}{4L^2} = \frac{1}{LC}$$

$$\therefore\ p^2 + \frac{R}{L}\cdot p + \frac{1}{LC} = p^2 + \frac{R}{L}\cdot p + \frac{R^2}{4L^2} = \left(p + \frac{R}{2L}\right)^2$$

$$\therefore\quad i = \frac{pE}{L\left(p + \frac{R^2}{2L}\right)^2}\ [1]$$

$$= \frac{E}{L}\cdot\frac{p}{\left(p + \frac{R}{2L}\right)^2}\ [1]$$

$$\frac{d\left(\frac{p}{p - R/2L}\right)}{d\left(\frac{R}{2L}\right)} = -\frac{p}{\left(p + \frac{R}{2L}\right)^2}$$

It has already been proved that

$$\frac{p}{p + a}\ [1] = e^{-at}$$

This is comparable with

$$\frac{p}{p + (R/2L)}\ [1] = e^{-R/2L\,.\,t}$$

Thus,

$$\frac{d}{d\,(R/2L)}\cdot\frac{p}{(p + (R/2L))}\ [1] = -\,t\,e^{-R/2L\,.\,t}$$

$$\therefore\quad \frac{p}{(p + (R/2L))^2} = +\ t\,e^{-R/2L\,.\,t}$$

$$\therefore\quad \frac{E}{L}\cdot\frac{p}{(p + (R/2L))^2} = \frac{E}{L}\,.\,t\,.\,e^{-R/2L\,.\,t}$$

It will be seen from the foregoing examples that solutions by Heaviside's calculus are simpler and more straightforward than by the ordinary calculus.

It may sometimes be necessary to evaluate the voltage expressions for various parts of the circuit. The method to be adopted will be illustrated below.

With reference to case (1) of the example just discussed :

$$i = \frac{E \, . \, e^{-R/2L\,.\,t}}{L\sqrt{\dfrac{R^2}{4\,L^2} - \dfrac{1}{L\,C}}} \sinh\left(\sqrt{\frac{R^2}{4\,L^2} - \frac{1}{L\,C}}\right) t$$

The voltage across the resistance $= Ri = e_R$

$$\therefore \quad e_R = \frac{R \, . \, E \, . \, e^{-R/2L\,.\,t}}{L\sqrt{\dfrac{R^2}{4\,L^2} - \dfrac{1}{L\,C}}} \sinh\left(\sqrt{\frac{R^2}{4\,L^2} - \frac{1}{L\,C}}\right) t$$

The voltage across the inductance $= L\, di/dt = e_L$.

$$\frac{di}{dt} = \frac{E \, . \, e^{-R/2\,L\,.\,t}}{L\sqrt{\dfrac{R^2}{4\,L^2} - \dfrac{1}{L\,C}}} \sqrt{\frac{R^2}{4\,L^2} - \frac{1}{L\,C}} \cosh\left(\sqrt{\frac{R^2}{4\,L^2} - \frac{1}{L\,C}}\right) t$$

$$- \frac{(R/2\,L)\; E \, . \, e^{-R/2L\,.\,t}}{\sqrt{\dfrac{R^2}{4\,L^2} - \dfrac{1}{L\,C}}} \sinh\left(\sqrt{\frac{R^2}{4\,L^2} - \frac{1}{L\,C}}\right) t$$

$$\therefore \quad L\frac{di}{dt} = \frac{L \, . \, E \, . \, e^{-R/2L\,.\,t}}{L\sqrt{\dfrac{R^2}{4\,L^2} - \dfrac{1}{L\,C}}} \,.\, \sqrt{\frac{R^2}{4\,L^2} - \frac{1}{L\,C}}$$

$$\cosh\left(\sqrt{\frac{R^2}{4\,L^2} - \frac{1}{L\,C}}\right) t$$

$$- \frac{L \, . \, R \, . \, E \, . \, e^{-R/2L\,.\,t}}{2\,L\sqrt{\dfrac{R^2}{4\,L^2} - \dfrac{1}{L\,C}}} \sinh\left(\sqrt{\frac{R^2}{4\,L^2} - \frac{1}{L\,C}}\right) t$$

$$= E\, e^{-R/2L\,.\,t} \cosh\left(\sqrt{\frac{R^2}{4\,L^2} - \frac{1}{L\,C}}\right) t - \frac{R \, . \, E \, . \, e^{-R/2L\,.\,t}}{2\,L\sqrt{\dfrac{R^2}{4\,L^2} - \dfrac{1}{L\,C}}}$$

$$\sinh\left(\sqrt{\frac{R^2}{L^2} - \frac{1}{L\,C}}\right) t$$

$$= E\, e^{-R/2L\,.\,t}\left[\cosh\left(\sqrt{\frac{R^2}{4L^2}-\frac{1}{LC}}\right)t - \frac{R}{2L\sqrt{\frac{R^2}{4L^2}-\frac{1}{LC}}}\right.$$

$$\left.\sinh\left(\sqrt{\frac{R^2}{4L^2}-\frac{1}{LC}}\right)t\right]$$

The voltage across the condenser may be found by evaluating the expression

$$\frac{1}{C}\int i\,.\,dt\left(=\frac{1}{C}\cdot\frac{1}{p}\,i\right)$$

or by re-arranging the voltage equation.

$$L\frac{di}{dt} + Ri + \frac{1}{C}\int i\,.\,dt = E$$

$$\therefore\ E - L\frac{di}{dt} - Ri = \frac{1}{C}\int i\,.\,dt$$

thus :

$$\frac{1}{C}\int i\,dt = E - \frac{R\,.\,E\,e^{-R/2L\,.\,t}}{\sqrt{\frac{R^2}{4L^2}-\frac{1}{LC}}}\sinh\left(\sqrt{\frac{R^2}{4L^2}-\frac{1}{LC}}\right)t$$

$$- E\,e^{-R/2L\,.\,t}\left[\cosh\left(\sqrt{\frac{R^2}{4L^2}-\frac{1}{LC}}\right)t - \frac{R}{2L\sqrt{\frac{R^2}{4L^2}-\frac{1}{LC}}}\right.$$

$$\left.\sinh\left(\sqrt{\frac{R^2}{4L^2}-\frac{1}{LC}}\right)t\right]$$

$$=E\left\{1 - \frac{R\,e^{-R/2L\,.\,t}}{L\sqrt{\frac{R^2}{4L^2}-\frac{1}{LC}}}\sinh\left(\sqrt{\frac{R^2}{4L^2}-\frac{1}{LC}}\right)t\right.$$

$$- e^{-R/2L\,.\,t}\cosh\left(\sqrt{\frac{R^2}{4L^2}-\frac{1}{LC}}\right)t$$

$$\left.+ \frac{R\,e^{-R/2L\,.\,t}}{2L\sqrt{\frac{R^2}{4L^2}-\frac{1}{LC}}}\sinh\left(\sqrt{\frac{R^2}{4L^2}-\frac{1}{LC}}\right)t\right\}$$

$$= \mathrm{E}\left\{1 - e^{-\mathrm{R}/2\mathrm{L}\,.\,t}\left[\frac{\mathrm{R}}{\mathrm{L}\sqrt{\frac{\mathrm{R}^2}{4\,\mathrm{L}^2} - \frac{1}{\mathrm{L\,C}}}} \sinh\left(\sqrt{\frac{\mathrm{R}^2}{4\,\mathrm{L}^2} - \frac{1}{\mathrm{L\,C}}}\right) t\right.\right.$$

$$+ \cosh\left(\sqrt{\frac{\mathrm{R}^2}{4\,\mathrm{L}^2} - \frac{1}{\mathrm{L\,C}}}\right) t$$

$$\left.\left. - \frac{\mathrm{R}}{2\,\mathrm{L}\sqrt{\frac{\mathrm{R}^2}{4\,\mathrm{L}^2} - \frac{1}{\mathrm{L\,C}}}} \sinh\left(\sqrt{\frac{\mathrm{R}^2}{4\,\mathrm{L}^2} - \frac{1}{\mathrm{L\,C}}}\right) t\right]\right\}$$

$$= \mathrm{E}\left\{1 - e^{-\mathrm{R}/2\,\mathrm{L}\,.\,t}\left[\cosh\left(\sqrt{\frac{\mathrm{R}^2}{4\,\mathrm{L}^2} - \frac{1}{\mathrm{L\,C}}}\right) t + \frac{\mathrm{R}}{\mathrm{L}\sqrt{\frac{\mathrm{R}^2}{4\,\mathrm{L}^2} - \frac{1}{\mathrm{L\,C}}}}\right.\right.$$

$$\left.\left.\left(\sinh\left(\sqrt{\frac{\mathrm{R}^2}{4\,\mathrm{L}^2} - \frac{1}{\mathrm{L\,C}}}\right) t - \tfrac{1}{2}\sinh\left(\sqrt{\frac{\mathrm{R}^2}{4\,\mathrm{L}^2} - \frac{1}{\mathrm{L\,C}}}\right) t\right)\right]\right\}$$

$$= \mathrm{E}\left\{1 - e^{-\mathrm{R}/2\,\mathrm{L}\,.\,t}\left[\cosh\left(\sqrt{\frac{\mathrm{R}^2}{4\,\mathrm{L}^2} - \frac{1}{\mathrm{L\,C}}}\right) t + \frac{\mathrm{R}}{2\mathrm{L}\sqrt{\frac{\mathrm{R}^2}{4\,\mathrm{L}^2} - \frac{1}{\mathrm{L\,C}}}}\right.\right.$$

$$\left.\left.\sinh\left(\sqrt{\frac{\mathrm{R}^2}{4\,\mathrm{L}^2} - \frac{1}{\mathrm{L\,C}}}\right) t\right]\right\}$$

In case (2) the voltage expressions are found in exactly the same manner. They are :

Voltage across resistance $= \mathrm{R}\,i = e_{\mathrm{R}}$

$$= \frac{\mathrm{R\,E}\,e^{-\mathrm{R}/2\,\mathrm{L}\,.\,t}}{\mathrm{L}\sqrt{\frac{1}{\mathrm{L\,C}} - \frac{\mathrm{R}^2}{4\,\mathrm{L}^2}}} \sin\left(\sqrt{\frac{1}{\mathrm{L\,C}} - \frac{\mathrm{R}^2}{4\,\mathrm{L}^2}}\right) t$$

Voltage across inductance $= \mathrm{L}\,di/dt = e_{\mathrm{L}}$

$$= \mathrm{E}\,e^{-\mathrm{R}/2\,\mathrm{L}\,.\,t}\left[\cos\left(\sqrt{\frac{1}{\mathrm{L\,C}} - \frac{\mathrm{R}^2}{4\,\mathrm{L}^2}}\right) t - \frac{\mathrm{R}}{2\,\mathrm{L}\sqrt{\frac{1}{\mathrm{L\,C}} - \frac{\mathrm{R}^2}{4\,\mathrm{L}^2}}}\right.$$

$$\left.\sin\left(\sqrt{\frac{1}{\mathrm{L\,C}} - \frac{\mathrm{R}^2}{4\,\mathrm{L}^2}}\right) t\right]$$

Voltage across the condenser $= \frac{1}{C} \int i \, dt = e_c$

$$= E \left\{ 1 - e^{-R/2L \cdot t} \left[\cos \left(\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}} \right) t + \frac{R}{2L\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}}} \sin \left(\sqrt{\frac{1}{LC} - \frac{R^2}{4L^2}} \right) t \right] \right\}$$

Similarly for case (3)

$$e_R = \frac{RE}{L} \cdot t \, e^{-R/2L \cdot t}$$

$$e_L = E \left[1 - \frac{Rt}{2L} \right] e^{-R/2L \cdot t}$$

$$e_C = E \left\{ 1 - \left[1 + \frac{Rt}{2L} \right] e^{-R/2L \cdot t} \right\}$$

It is known that the energy stored in an electrostatic field (capacitance) is of the form, $W = \frac{1}{2} C V^2$

The corresponding expression for the energy stored in a magnetic field (inductance) is, $W = \frac{1}{2} L I^2$

Therefore it is possible to evaluate these energy expressions.

Table I. summarises in detail the various expressions derived from the foregoing discussion on the R.L.C. circuit.

The time/current voltage characteristics for each case are shown in Fig. 12.

It has already been shown that real equations may have complex roots. Thus :

$$\phi\,(m + jn) = x + jy,$$

where j only is unreal.

Substituting in a typical quadratic type of equation,

$$\begin{aligned} & p\,(m+jn)^2 + q\,(m+jn) + r \\ &= pm^2 - pn^2 + 2pjmn + qm + qjn + r \\ &= p\,(m^2 - n^2) + qm + j\,(2pmn + qn) + r \end{aligned}$$

Similarly,

$$\begin{aligned} & p\,(m-jn)^2 + q\,(m-jn) + r \\ &= pm^2 - pn^2 - 2jpmn + qm - qjn + r \\ &= p\,(m^2 - n^2) + qm - j\,(2pmn + qn) + r \end{aligned}$$

$$\therefore \quad \phi\,(m - jn) = x - jy.$$

TABLE I.

CIRCUIT — RLC. — SERIES

Type of Root.	i	e_R	e_L	e_C	$P_{magnetic.}$	$P_{electrostatic.}$	Expansion Theorem Used.
$\frac{R^2}{4L^2} > \frac{1}{LC}.$	$\frac{E e^{-\frac{R}{2L}t}}{L\sqrt{\frac{R^2}{4L^2}-\frac{1}{LC}}}\left[\sinh\left(\sqrt{\frac{R^2}{4L^2}-\frac{1}{LC}}\right)t\right].$	$\frac{RE e^{-\frac{R}{2L}t}}{L\sqrt{\frac{R^2}{4L^2}-\frac{1}{LC}}}\left[\sinh.\left(\sqrt{\frac{R^2}{4L^2}-\frac{1}{LC}}\right)t\right]$	$E.e^{-\frac{R}{2L}t}\left[\cosh\left(\sqrt{\frac{R^2}{4L^2}-\frac{1}{LC}}\right)t - \frac{R}{2L\sqrt{\frac{R^2}{4L^2}-\frac{1}{LC}}}\sinh\left(\sqrt{\frac{R^2}{4L^2}-\frac{1}{LC}}\right)t\right]$	$E\left\{1-e^{-\frac{R}{2L}t}\left[\cosh.\left(\sqrt{\frac{R^2}{4L^2}-\frac{1}{LC}}\right)t + \frac{R}{2L\sqrt{\frac{R^2}{4L^2}-\frac{1}{LC}}}\sinh.\left(\sqrt{\frac{R^2}{4L^2}-\frac{1}{LC}}\right)t\right]\right\}$	$\frac{E^2\,2LC}{R^2C-4L}.e^{-\frac{R}{L}t}\sinh^2\left(\sqrt{\frac{R^2}{4L^2}-\frac{1}{LC}}\right)t.$	$\frac{E^2}{2}C\left\{1-e^{-\frac{R}{2L}t}\left[\cosh.\left(\sqrt{\frac{R^2}{4L^2}-\frac{1}{LC}}\right)t + \frac{R}{2L\sqrt{\frac{R^2}{4L^2}-\frac{1}{LC}}}\sinh.\left(\sqrt{\frac{R^2}{4L^2}-\frac{1}{LC}}\right)t\right]\right\}^2$	YES.
$\frac{R^2}{4L^2} < \frac{1}{LC}$	$\frac{E}{L\sqrt{\frac{1}{LC}-\frac{R^2}{4L^2}}}e^{-\frac{R}{2L}t}\sin.\left(\sqrt{\frac{1}{LC}-\frac{R^2}{4L^2}}\right)t$	$\frac{RE e^{-\frac{R}{2L}t}}{L\sqrt{\frac{1}{LC}-\frac{R^2}{4L^2}}}\sin.\left(\sqrt{\frac{1}{LC}-\frac{R^2}{4L^2}}\right)t$	$E\,e^{-\frac{R}{2L}t}\left[\cos.\left(\sqrt{\frac{1}{LC}-\frac{R^2}{4L^2}}\right)t - \frac{R}{2L\sqrt{\frac{1}{LC}-\frac{R^2}{4L^2}}}\sin.\left(\sqrt{\frac{1}{LC}-\frac{R^2}{4L^2}}\right)t\right]$	$E\left\{1-e^{-\frac{R}{2L}t}\left[\cos.\left(\sqrt{\frac{1}{LC}-\frac{R^2}{4L^2}}\right)t + \frac{R}{2L\sqrt{\frac{1}{LC}-\frac{R^2}{4L^2}}}\sin.\left(\sqrt{\frac{1}{LC}-\frac{R^2}{4L^2}}\right)t\right]\right\}$	$\frac{E^2\,2LC}{4L-R^2C}\,e^{-\frac{R}{L}t}\sin^2\left(\sqrt{\frac{1}{LC}-\frac{R^2}{4L^2}}\right)t$	$\frac{E^2}{2}C\left\{1-e^{-\frac{R}{2L}t}\left[\cos.\left(\sqrt{\frac{1}{LC}-\frac{R^2}{4L^2}}\right)t + \frac{R}{2L\sqrt{\frac{1}{LC}-\frac{R^2}{4L^2}}}\sin.\left(\sqrt{\frac{1}{LC}-\frac{R^2}{4L^2}}\right)t\right]\right\}^2$	YES.
$\frac{R^2}{4L^2} = \frac{1}{LC}$	$\frac{E}{L}.t\,e^{-\frac{R}{2L}t}$	$\frac{RE}{L}t\,e^{-\frac{R}{2L}t}$	$E\left[1-\frac{Rt}{2L}\right]e^{-\frac{R}{2L}t}$	$E\left\{1-\left[1+\frac{Rt}{2L}\right]e^{-\frac{R}{2L}t}\right\}$	$\frac{E^2}{2}\frac{1}{L}t^2 e^{-\frac{R}{L}t}$	$\frac{E^2}{2}C\left\{1-\left[1+\frac{Rt}{2L}\right]e^{-\frac{R}{2L}t}\right\}^2$	NO.

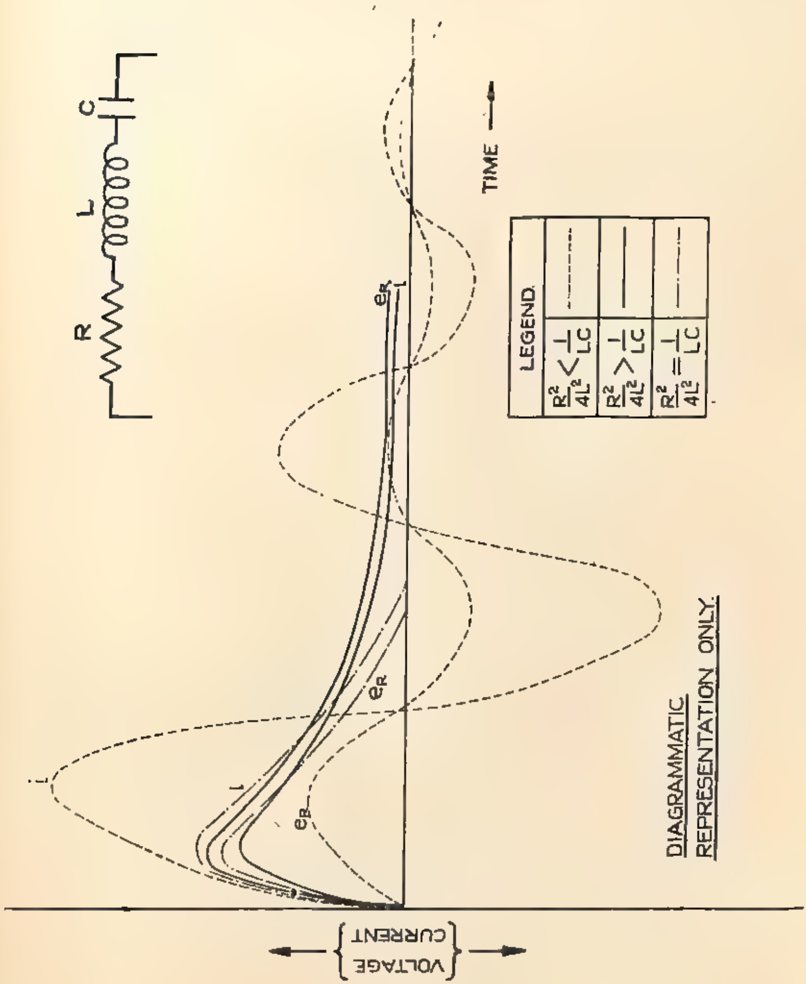


Fig 12.—Characteristics from R.L.C. circuit.

Simple Substitution in the Expansion Theorem, when Roots are Complex.

From A.C. theory it is known that vectors can be represented symbolically. The following relationships may be defined :

Reference to Fig. 13.

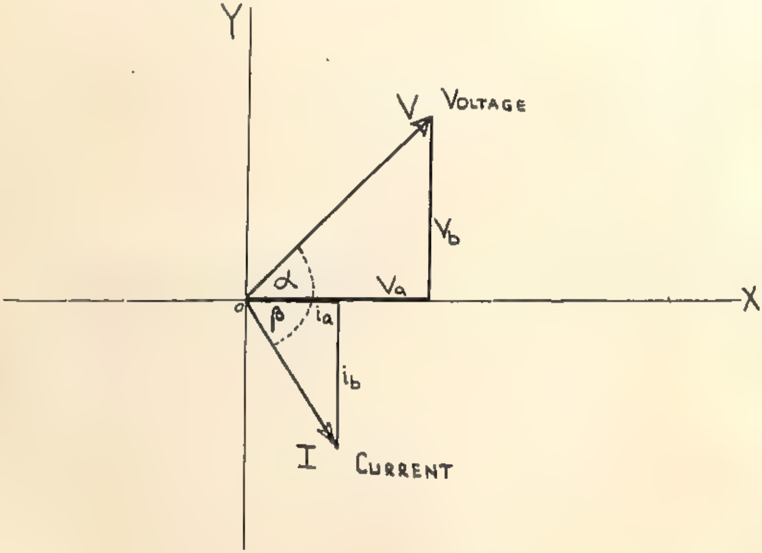


Fig. 13.—Symbolic representation of vectors.

Voltage V may be expressed $V\underline{/\alpha}$ and current I. may be expressed $I\overline{/\beta}$.

$\underline{/\alpha}$ means that the vector V is displaced from the X-axis by an angle α.

$\overline{/\beta}$ means that the vector I. is displaced by an angle β from the X – axis.

Vector V may be expressed also :

$\underset{\cdot}{V} = V_a + j\,V_b$, where V_a and V_b are referred to rectangular cartesian axes OX, OY. and current $\underset{\cdot}{I} = i_a - ji_b$

They may also have the form :

$$V\underline{/\alpha} = \underset{\cdot}{V} = V\cos\alpha + j\,V\sin\alpha.$$

$$I\overline{/\beta} = \underset{\cdot}{I} = I\cos\beta - j\,I\sin\beta.$$

Taking a more general case, Fig. 14 illustrates an example similar to Fig. 13.

$x + jy$ may be regarded as being represented by the point Z, or as being represented by the rotation of the vector OZ through an angle θ.

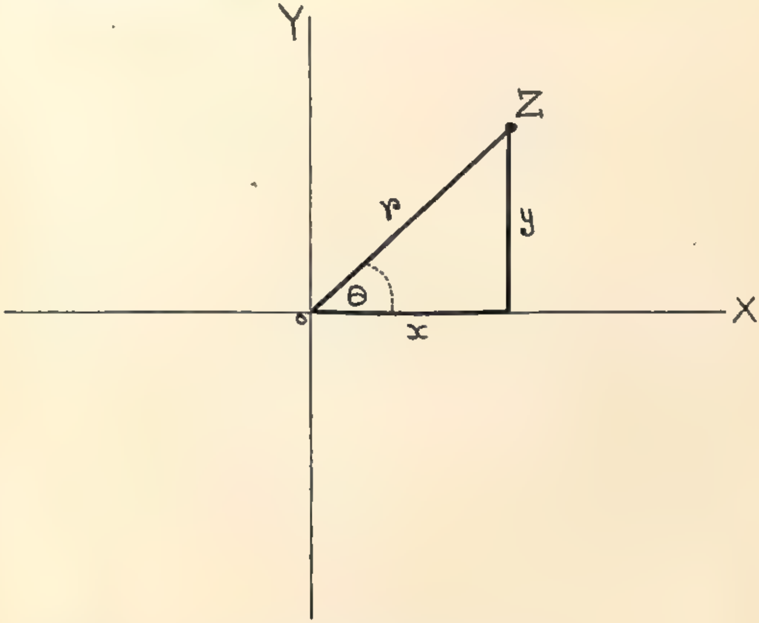


Fig. 14.—Argand Diagram.

Referring to polar co-ordinates $r\ \theta$,

$$x + jy = Z = r\,[\cos\theta + j\sin\theta]\ \ (r \geq 0\,;\ -\pi < \theta \leq \pi).$$

The modulus of $(x + jy) = r$

$$\therefore\quad |Z| = |x + jy| \text{ and } \theta = arg\,Z$$

It is known that if $y = A\,e^{bx}$

$$\frac{dy}{dx} = b\,A\,e^{bx} = by.$$

Consider the expression, $Z = \cos\theta + j\sin\theta$.

$$\frac{dz}{d\theta} = -\sin\theta + j\cos\theta.$$

$$= j\,[\cos\theta + j\sin\theta]$$

$$\therefore\quad \frac{dz}{d\theta} = j\,[Z]$$

By comparison, $Z = Ae^{j\theta}$

$$\begin{array}{lll} \text{when} & \theta & = 0 \\ & Z & = \cos 0 + j \sin 0 \\ \therefore & Z & = 1 \\ \text{thus} & I & = A\, e^{j0} \qquad \therefore \quad A = 1. \\ \therefore & Z & = \cos \theta + j \sin \theta = e^{j\theta} \end{array}$$

The following identities exist :

$$\cos \theta + j \sin \theta \equiv e^{j\theta}$$

$$\therefore \quad r\,[\cos \theta + j \sin \theta] \equiv r\, e^{j\theta} \equiv x + jy$$

In general it may be written :

$$a \sqrt{x+jy} = |a \sqrt{r}| \left[\cos \frac{\theta + 2n\pi}{a} + j \sin \frac{\theta + 2\pi n}{a} \right]$$

$$= |a\sqrt{r}|\, e^{\frac{j(\theta+2n\pi)}{a}}$$

where n is an integer.

If the complex expression for $p = -\alpha + j\beta$ and the vectorial representation $= r + jx$ corresponding to $Ae^{j\theta}$, then $A = \sqrt{r^2+x^2}$ and the argument of $A = \tan^{-1} r/x$.

Similarly $Y = r - jx = A\, e^{-j\theta}$

$\therefore$ We now have :

$$A\,[e^{j\theta}\, e^{(-\alpha+j\beta)t} + e^{-j\theta}\, e^{(-\alpha-j\beta)t}]$$

$$\left(\cos \phi = \frac{e^{j\phi} + e^{-j\phi}}{2} \right)$$

$$= 2\,A\,e^{-\alpha t} \cos (pt + \theta)$$

This expression will now be substituted in the expansion theorem in place of

$$\frac{Y_{(p)}}{p\, dZ_{(p)}/dp}$$

where p is expressed as above.

Cubic Equations.

Let the general form of the equation be :

$$\alpha x^3 + \beta x^2 + \delta x + \lambda = 0 \tag{44}$$

Reducing this equation in the usual manner (*i.e.*, substituting for x the expression $\left(x - \frac{\beta}{\alpha} \cdot \frac{1}{3}\right)$

If the expression was of the form :

$$x^3 + \beta x^2 + \delta x + \lambda = 0 \qquad (45)$$

the substitution would be $(x - \beta/3)$.

With reference to equation (45) the expression becomes :

$$\left[x^3 - \beta x^2 + \frac{\beta^2 x}{3} - \frac{\beta^3}{27}\right] + \beta\left[x^2 - \frac{2\beta x}{3} + \frac{\beta^2}{9}\right]$$

$$+ \delta\left[x - \frac{\beta}{3}\right] + \lambda = 0$$

$$= x^3 + x\left[\delta - \frac{\beta^2}{3}\right] + \frac{2}{27}\beta^3 - \tfrac{1}{3}\delta\beta + \lambda = 0$$

This is comparable with

$$x^3 + mx + n = 0 \qquad (46)$$

($i.e.$, without a term in x^2).

The Method of solving a Cubic Equation (due to Cardan).

Let $\quad x = a + b$

$\therefore \quad x^3 = a^3 + 3a^2b + 3ab^2 + b^3$

which, by substitution, may be written :

$$a^3 + b^3 + 3abx \qquad (47)$$

If $\quad 3ab = -m \qquad (47a)$

and $\quad a^3 + b^3 = -n \qquad (47b)$

Equations 46 and (47) be similar.

From equation (47a) :

$$ab = -\frac{m}{3} \quad \text{or} \quad a^3b^3 = -\frac{m^3}{27}$$

Consider the quadratic :

$$\theta^2 + n\theta - \frac{m^3}{27}$$

$$\theta = \frac{-n \pm \sqrt{n^2 + 4m^3/27}}{2}$$

$$= -\frac{n}{2} \pm \sqrt{\frac{n^2}{4} + \frac{m^3}{27}}$$

a^3 and b^3 are roots of the equation.

$$\therefore \quad x = a + b = \left[-\frac{n}{2} + \left(\frac{n^2}{4} + \frac{m^3}{27} \right) \right]^{\frac{1}{3}}$$
$$+ \left[-\frac{n}{2} - \left(\frac{n^2}{4} + \frac{m^3}{27} \right) \right]^{\frac{1}{3}}$$

So far it has been seen that for a linear equation

$$(i.e., \quad ax + b = 0\,; \quad x = -b/a)$$

there is one root only.

For a quadratic equation

$$\left(i.e., \quad ax^2 + bx + c = 0, \; x = \frac{-b \pm \sqrt{b^2 - 4\,ac}}{2}\right)$$

there are two roots.

For a cubic equation, therefore, there will be three roots.

Consider the equation :

$$x^3 = 1 \quad (i.e., \; x^3 - 1 = 0),$$

and let the roots be : $1, \; a, \; a^2$.

The operator j or i $(= \sqrt{-1})$, has already been discussed. This will now be enlarged slightly. Referring to Fig. 15, this gives a composite picture of operator j and operator a, the latter being a root of the cubic equation under discussion. This has the effect of turning a vector through $2\,\pi/3$ radians (120°).

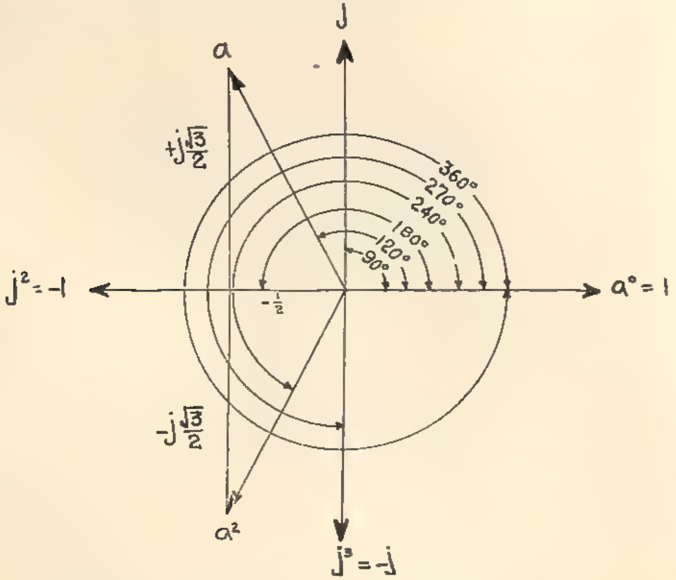


Fig. 15.—Illustration of operator's "a" and "j:"

Writing symbolically,

$$a = -\tfrac{1}{2} + j\frac{\sqrt{3}}{2}$$

$$= \cos\frac{2\pi}{3} + j\sin\frac{2\pi}{3} = e^{j(2\pi/3)}$$

$$a^2 = -\tfrac{1}{2} - j\sqrt{3}/2$$

$$\therefore \quad a^3 = 1 \text{ (i.e., } 2\pi \text{ radians} = 360°\text{)}.$$

Thus, $1 + a + a^2 = 0$.

From the geometry of the figure, it is possible to add two intermediate functions of a, namely :

$$a^{\frac{1}{2}} = 60° \text{ shift.}$$
$$a^{-\frac{1}{2}} = 300° \text{ shift.}$$

Expressing j in terms of a :

$$a = -\tfrac{1}{2} + j\frac{\sqrt{3}}{2} \tag{48}$$

$$a^2 = -\tfrac{1}{2} - j\frac{\sqrt{3}}{2} \tag{49}$$

$$j = \frac{1}{\sqrt{3}} + \frac{2a}{\sqrt{3}} \tag{48a}$$

This operator a is used in the analysis of unsymmetrical three-phase circuits.

In the case of $\dfrac{n^2}{4} + \dfrac{m^3}{27}$ being positive,

i.e., $27n^2 > 4m^3$, there are two complex roots ; and one real root. They are of the form,

(I.) $a + b$

(II.) $\dfrac{-(a+b)}{2} + j\dfrac{\sqrt{3}}{2}(a-b)$

(III.) $\dfrac{-(a+b)}{2} - j\dfrac{\sqrt{3}}{2}\cdot(a-b)$

In the case of $n^2/4 + m^3/27$ being negative, *i.e.*, $27n^2 < 4m^3$ there are three real roots a and b have complex conjugate roots of the form,

$$a = \alpha + j\beta$$
$$a = \alpha - j\beta$$

Then the roots are :

(I.) $-\alpha + \sqrt{3}\,\beta$
(II.) $-\alpha - \sqrt{3}\,\beta$
(III.) $2\,\alpha$

Lastly is the case of equal roots, *i.e.*, $\frac{n^2}{4} + \frac{m^3}{27} = 0$, all three roots are real and at least two are equal.

They are : (I) $2\,a$
(II.) $-a$
(III.) $-a$

In the second case considered,

$$\frac{n^2}{4} + \frac{m^3}{27} < 0,$$

the three real roots yielded, require the extraction of cube roots of complex quantities for a numerical solution. Thus, in its algebraic form, the expression is irreducible, therefore trigonometrical solution is required.

The Quartic Equation.

As the degree of an equation becomes higher, more specialised solutions are required. The highest degree of equation which will be discussed in this work is that of the fourth power in x, namely the quartic or biquadratic equation.

Method of Solving a Quartic Equation.

A quartic equation may be of the form :

$$x^4 + a\,x^3 + b\,x^2 + c\,x + d = 0$$

This may be reduced to a pair of quadratic equations by finding a real root, y, of the cubic equation :

$$8y^3 - 4\,by^2 + 2\,(ac - 4\,d)\,y - [c^2 + d\,(a^2 - 4b)\,] = 0$$

Thus four roots may be extracted by solving the two following quadratic equations :

$$x^2 + \left[\frac{a}{2} - \sqrt{\frac{a^2}{4} + 2y_1 - b}\right] x + (y_1 + \sqrt{y_1^2 - d}) = 0 \qquad (50)$$

$$x^2 + \left[\frac{a}{2} + \sqrt{\frac{a^2}{4} + 2y_1 - b}\right] x + (y_1 - \sqrt{y_1^2 - d}) = 0 \qquad (51)$$

The roots are :

$$\tfrac{1}{2}\left\{-\left[\frac{a}{2}-\sqrt{\frac{a^2}{4}+2y_1-b}\right]\right.$$

$$\left.\pm\sqrt{\left[\frac{a}{2}-\sqrt{\frac{a^2}{4}+2y_1-b}\right]^2-4\left[y_1+\sqrt{y_1{}^2-d}\right]}\right\} \quad (52)'$$

$$\tfrac{1}{2}\left\{-\left[\frac{a}{2}+\sqrt{\frac{a^2}{4}+2y_1-b}\right]\right.$$

$$\left.\pm\sqrt{\left[\frac{a}{2}+\sqrt{\frac{a^2}{4}+2y_1-b}\right]^2-4\left[y_1-\sqrt{y_1{}^2-d}\right]}\right\} \quad (53)$$

THE SHIFTING OPERATION.

From the theory of the calculus it can be proved that :

$$\mathrm{D}\, e^{\alpha t}\,\mathrm{A} \quad (\mathrm{A} = \phi\,(t)) \qquad (54)$$

$$= e^{\alpha t}\,\mathrm{DA} + \mathrm{A}\,\alpha\, e^{\alpha t}$$

$$= e^{\alpha t}\,[\mathrm{D}+\alpha]\,\mathrm{A} \qquad (55)$$

Similarly, $\mathrm{D}^2\, e^{\alpha t}\,\mathrm{A}$

$$= e^{\alpha t}\,\mathrm{D}\,[\mathrm{D}+\alpha]\,\mathrm{A} + \alpha\, e^{\alpha t}\,[\mathrm{D}+\alpha]\,\mathrm{A}$$

$$= e^{\alpha t}\,[\mathrm{D}+\alpha]\,[\mathrm{D}+\alpha]\cdot\mathrm{A}$$

Thus in general terms :

$$\mathrm{D}^n\, e^{\alpha t}\,\mathrm{A} = e^{\alpha t}\,[\mathrm{D}+\alpha]^n\,\mathrm{A}$$

also, $\Phi\,(\mathrm{D})^n\, e^{\alpha t}\,\mathrm{A} \quad = e^{\alpha t}\,\Phi\,(\mathrm{D}+\alpha)^n\,\mathrm{A}$ (56)

Heaviside writes his shifting operation (reference "Electromagnetic Theory") as :

$$\mathrm{P}\,[\lambda\, e^{-\alpha t}] = e^{-\alpha t}\,\frac{d\lambda}{dt} - e^{-\alpha t}\,\alpha\,\lambda$$

$$= e^{-\alpha t}\,[p-\alpha]\,\lambda \qquad (57)$$

It is seen that when the operator p is transferred to the right of the exponential function $e^{-\alpha t}$ it (p) becomes $(p-\alpha)$. The reverse of this process transforms p into $(p+\alpha)$.

Comparison of Laplacian Transformations with Heaviside's Methods.

The Laplacian Direct Transformation is written :

$$\mathrm{L}\, f(t) = \int_0^\infty e^{-pt} f(t)\, dt \quad \left(\begin{array}{l} p > 0 \neq 0 \\ f(t) = 0 \text{ when } t < 0 \end{array}\right\}$$

$$\int_0^\infty e^{-pt} f(t)\, dt = \phi(p)$$

$$\therefore \quad \text{L}\, f(t) = \phi(p)$$

Inverse Transformation.

The direct transformation may be inverted,

$$f(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{tp} f(p)\, dp \qquad (c > 0)$$

$$\therefore \quad \text{L}^{-1}\, \phi(p) = f(t)$$

(*Note.*—The function undergoing transformation is sometimes denoted $\widetilde{f(t)}$).

Transformation of unit function,

$$\widetilde{\phi(p)} = \int_0^\infty e^{-pt} f(t)\, dt$$

$$= -\left[\frac{1}{p} e^{-pt}\right]_0^\infty = \frac{1}{p}$$

Thus Heaviside's operation is the reverse of the Laplacian operation.

Let λ be a function of t, $f(t)$
and δ also be a function of t, $\phi(t)$.

Let the term $f(t)\ [1]\ \phi(t)$ be some point on a wave expressing a function at a particular time interval.

Performing a Laplacian transformation on $\text{F}(\lambda)\ [1]\ (\delta)$, we have,

$$\text{L}\ \text{F}(\lambda)\ [1]\ (\delta) = \int_0^\infty e^{-pt}\, \text{F}(\lambda)\ [1]\ (\delta)\, dt \qquad (58)$$

Referring the time interval to a new set of Cartesian ordinates $O'x$, $O'y$, where it is understood the previous ordinates were Ox, Oy. The new time-interval is $t_1 = t - \beta$.

Thus equation (58) becomes :

$$\int_0^\infty e^{-p(t+\beta)}\ \text{F}(\lambda_{t_1})\ [1]\ (\delta_{t_1})\, d t_1$$

$$= e^{-p\beta}\int_0^\infty e^{-pt}\ \text{F}(t_1)\, d t_1$$

By definition, $\widetilde{\phi(p)} = \int_0^\infty e^{-pt} f(t)\, dt$

$$\therefore \int_0^\infty e^{-p(t+\beta)} \; F(\lambda t_1)\,[1]\,(\delta t_1)\, d t_1 = e^{-p\beta}\, \widetilde{\phi(p)}$$

The factor $\varepsilon^{-p\beta}$ being the shifting operator.

METHOD OF TREATING ALTERNATING CURRENTS.

By definition, $\sin x = \dfrac{e^{jx} - e^{-jx}}{2j}$

$$= \frac{1}{2j}\left[e^{jx} - e^{-jx} \right]$$

It has been shown that

$$\frac{p}{p+a}[1] = e^{-at}[1]$$

$$\text{and } \frac{p}{p-a}[1] = e^{at}[1]$$

$$\text{Thus } \frac{p}{p \pm a}[1] = e^{\mp at}[1]$$

$$\therefore \quad \sin xt\,[1] = \frac{e^{+jxt} - e^{-jxt}}{2j}\,[1]$$

$$= \frac{1}{2j}\left[\frac{p}{p-jx}[1] - \frac{p}{p+jx}[1] \right]$$

$$= \frac{1}{2j}\left[\frac{p(p+jx) - p(p-jx)}{(p-jx)(p+jx)} \right][1]$$

$$= \frac{p}{2j}\left[\frac{p + jx - p + jx}{p^2 + x^2} \right][1] = \frac{p}{2j}\left[\frac{2jx}{p^2+x^2} \right][1]$$

$$= \frac{px}{p^2+x^2}[1] \tag{59}$$

Similarly, $\cos x = \dfrac{e^{jx} + e^{-jx}}{2}$

$$\therefore \quad \cos xt \; [1] = \tfrac{1}{2} \left[\frac{p}{p - jx} \, [1] + \frac{p}{p + jx} \, [1] \right]$$

$$= \tfrac{1}{2} \left[\frac{p\,(p + jx) + p\,(p - jx)}{p^2 + x^2} \right] [1]$$

$$= \frac{p}{2} \left[\frac{p + jx + p - jx}{p^2 + x^2} \right] [1]$$

$$= \frac{p}{2} \left[\frac{2p}{p^2 + x^2} \right] [1]$$

$$= \frac{p^2}{p^2 + x^2} \; [1] \qquad (60)$$

A more familiar form of alternating voltage is $\sin(\omega t + \phi)$.

$$\sin(\omega t + \phi) \; [1] = [\sin \omega t \cos \phi + \cos \omega t \sin \phi] \; [1]$$

$$\sin w t = \frac{p\,\omega}{p^2 + \omega^2}; \qquad \cos \phi = \tfrac{1}{2} \left[e^{j\phi} + e^{-j\phi} \right]$$

$$\cos w t = \frac{p^2}{p^2 + \omega^2}; \qquad \sin \phi = \frac{1}{2j} \left[e^{j\phi} - e^{-j\phi} \right]$$

$$\therefore \sin(\omega t + \phi \; [1] = \left[\frac{p\,\omega}{p^2 + \omega^2} \cdot \left(\frac{e^{j\phi}}{2} + \frac{e^{-j\phi}}{2} \right) + \frac{p^2}{p^2 + \omega^2} \left(\frac{e^{j\phi}}{2j} - \frac{e^{-j\phi}}{2j} \right) \right] [1]$$

$$= \frac{p\,\omega\,(\cos \phi) + p^2\,(\sin \phi)}{p^2 + w^2} \; [1]$$

Similarly :

$$\cos(w t \mp \phi) \; [1] = \frac{p^2\,(\cos \phi) \pm p\,w\,(\sin \phi)}{p^2 + \omega^2} \; [1]$$

When using the foregoing methods it is sometimes found that the denominator of the expansion theorem is in the form of a product.

When this contingency arises, the following simplification may be employed :

Let $Z_{(p)}$ be a product of the form : $f_{(p)} \cdot F_{(p)}$

The roots of $f_{(p)} = 0$, are $a.\ b,\ c.$
The roots of $F_{(p)} = 0$, are $\alpha,\ \beta,\ \lambda.$

$$\begin{aligned} Z'(a) &= f'(a) \cdot F(a) \\ Z'(b) &= f'(b) \cdot F(b) \\ Z'(c) &= f'(c) \cdot F(c), \quad \text{etc.} \end{aligned}$$

$$\text{Since} \left.\begin{array}{l} f(a) \\ f(b) \\ f(c) \text{ etc.} \end{array}\right\} = 0$$

$$\begin{aligned} Z'(\alpha) &= f(\alpha)\ F'(\alpha) \\ Z'(\beta) &= f(\beta)\ F'(\beta) \\ Z'(\lambda) &= f(\lambda)\ F'(\lambda) \text{ etc.} \end{aligned}$$

$$\text{Since} \left.\begin{array}{l} F(\alpha) \\ F(\beta) \\ F(\lambda) \text{ etc.} \end{array}\right\} = 0$$

Then

$$i = E\left[\frac{Y_{(0)}}{Z_{(0)}} + \left(\frac{Y_{(a)}\ e^{at}}{a\, f'_{(a)}\ F_{(a)}} + \frac{Y_{(b)}\ e^{bt}}{b\, f'_{(b)}\ F_{(b)}} + \frac{Y_{(c)}\ e^{ct}}{c\, f'_{(c)}\ F_{(c)}} + \frac{Y_{(\alpha)}\ e^{\alpha t}}{\alpha\, f_{(\alpha)}\ F'_{(\alpha)}} + \frac{Y_{(\beta)}\ e^{\beta t}}{\beta\, f_{(\beta)}\ F'_{(\beta)}} + \frac{Y_{(\lambda)}\ e^{\lambda t}}{\lambda\, f_{(\lambda)}\ F'_{(\lambda)}}\right)\right] \quad [1]$$

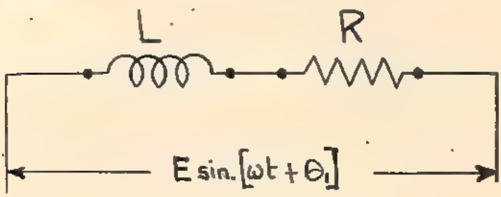


Fig. 16.—R.L. Circuit.

Consider a resistance and inductance in series, across which an alternating e.m.f. of the form $E \sin(\omega t + \theta_1)$, is applied. See Fig. 16.

The operational solution is of the form

$$i = \frac{E}{R + pL} \cdot \frac{p\omega \cos\theta_1 + p^2 \sin\theta_1}{p^2 + \omega^2}$$

$$\begin{aligned} Y_{(p)} &= p\omega \cos\theta_1 + p^2 \sin\theta_1 \\ Z_{(p)} &= (R + pL)\ (p^2 + \omega^2)) \end{aligned}$$

Comparable with $f_{(p)}$, $F_{(p)}$,

Roots of $f_{(p)} = 0$, are

$$p_1 = -R/L \quad (a = p_1 = p)$$

Roots of $F_{(p)} = 0$, are

$$p_2 = +j\omega \quad (a = p_2 = p)$$

$$p_3 = -j\omega \quad (\beta = p_3 = p)$$

$$\begin{aligned} f_{(p)} &= r + pL \\ p\,f'_{(p)} &= pL \\ F_{(p)} &= p^2 + \omega^2 \\ p\,F'_{(p)} &= 2p^2 \end{aligned}$$

$$\begin{aligned} \therefore \quad p\,f'_{(p)}\;F_{(p)} &= pL\,(p^2 + \omega^2) \\ p\,f_{(p)}\;F'_{(p)} &= 2p^2\,(r + pL) \end{aligned}$$

Substituting for p_1:

$$\begin{aligned} pL\,(p^2 + \omega^2) &= -\frac{rL}{L}\left[\frac{r^2}{L^2} + w^2\right] \\ &= -R\left[\frac{r^2 + \omega^2 L^2}{L^2}\right] \\ &= -\frac{R}{L^2}\left[r^2 + \omega^2 L^2\right] \end{aligned}$$

From Fig. 17, $\omega L/R = \tan^{-1}\theta_2$

Substituting in $Y_{(p)}$

$$\begin{aligned} &= -\frac{R\omega}{L}\cos\theta_1 + \frac{R^2}{L^2}\sin\theta_1 \\ &= -\frac{LR\omega\,.\cos\theta_1 + R^2\sin\theta_1}{L^2} \\ &= \frac{R\,[R\sin\theta_1 - L\omega\cos.\theta_1]}{L^2} \\ &= \frac{R}{L^2}\,[R\,.\sin\theta_1 - L\omega\,.\cos\theta_1] \end{aligned}$$

$$\frac{R}{\sqrt{R^2 + \omega^2 L^2}} = \cos\theta_2$$

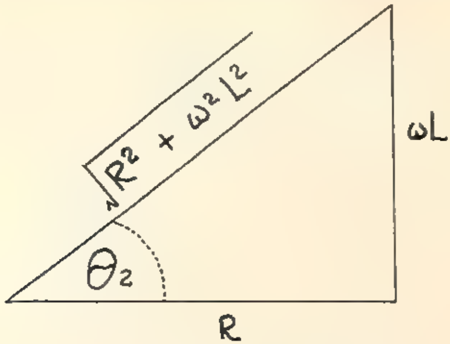


Fig. 17.—Pertaining to R.L. circuit.

$$\frac{\omega L}{\sqrt{R^2 + \omega^2 L^2}} = \sin \theta_2$$

$$R = \sqrt{R^2 + \omega^2 L^2} \cos \theta_2$$
$$\omega L = \sqrt{R^2 + \omega^2 L^2} \sin \theta_2$$

$$= \frac{R}{L^2} [\sqrt{R^2 + \omega^2 L^2} \cos \theta_2 \sin \theta_1$$

$$- \sqrt{R^2 - \omega^2 L^2} . \sin \theta_2 \cos \theta_1]$$

$$= \frac{R}{L^2} [(\sqrt{R^2 + \omega^2 L^2}) \sin (\theta_1 - \theta_2)]$$

The first part of the simplified expansion theorem may be written :

$$\frac{R/L^2 [(\sqrt{R^2 + \omega^2 L^2}) \sin (\theta_1 - \theta_2)]}{- R/L^2 [R^2 + \omega^2 L^2]} . E . e^{-R/L.t}$$

$$= - \frac{E . e^{-R/L.t} \sin (\theta_1 - \theta_2}{\sqrt{R^2 + \omega^2 L^2}}$$

The next root can be dealt with in a similar manner. The solution is well-known and is of the form :

$$i = \frac{E}{\sqrt{R^2 + \omega^2 L^2}} [\sin (\omega t + \theta_1 - \theta_2) - e^{-R/L.t} \sin (\theta_1 - \theta_2)] \tag{61}$$

To some readers this process may seem rather involved, although actually each step is straightforward. However, for the benefit

of those readers, the same circuit will be solved by the ordinary calculus in an abbreviated form.

Firstly, the voltage equation is established, and is of the form :

$$R\,i + L\,\frac{di}{dt} = E \,.\, \sin \omega t.$$

The complementary function gives the transient part of the solution, and is :

$$(R + L\,D)\,i = 0$$

$$\therefore \qquad i = A\,e^{-R/L\,.\,t}$$

$$\frac{1}{L}\,.\,R\,i + \frac{di}{dt} = \frac{1}{L}\,.\,E \sin \omega t$$

$$\left(\frac{1}{L}\,.\,R + \frac{d}{dt}\right) i = \frac{1}{L}\,.\,E \sin \omega t$$

$$i = \frac{\frac{1}{L}\,.\,E \sin \omega t}{\frac{1}{L}\,.\,R + D} \qquad (D = d/dt)$$

sin ωt may be written in the form : e^{jwt}

$$\therefore \qquad i = \frac{E/L\ e^{jwt}}{R/L + D} = \frac{E/L\ e^{jwt}}{R/L + j\omega}$$

Rationalising and discarding imaginaries,

$$= \frac{\frac{R}{L}\,.\,\frac{E}{L} \sin \omega t - \frac{\omega E}{L} \cos wt}{R^2/L^2 + \omega^2}$$

$$= \frac{\frac{R.E}{L^2} \sin \omega t - \frac{\omega E}{L} \cos \omega t}{\frac{R^2 + \omega^2 L^2}{L^2}}$$

$$= \frac{R\,E \sin \omega t - L\,w\,E \cos \omega t}{R^2 + \omega^2 L^2} = \frac{E\,(R \sin \omega t - L\,\omega \cos \omega t)}{R^2 + \omega^2 L^2}$$

$$= \frac{E}{\sqrt{R^2 + \omega^2 L^2}} \cdot \frac{R \sin \omega t - L\,\omega \cos \omega t}{\sqrt{R^2 + \omega^2 L^2}}$$

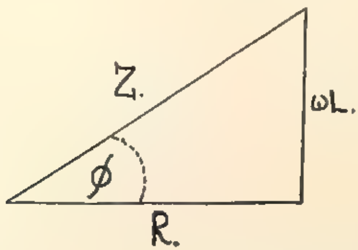


Fig. 18.—Pertaining to R.L. circuit (alternative method).

From Fig. 18 :

$$\left.\begin{aligned} \cos \phi &= \frac{R}{\sqrt{R^2 + \omega^2 L^2}} \\ \sin \phi &= \frac{\omega L}{\sqrt{R^2 + \omega^2 L^2}} \end{aligned}\right\}$$

$$\therefore \quad i = \frac{E \quad \cos \phi \sin \omega t - \sin \phi \cos \omega t}{\sqrt{R^2 + \omega^2 L^2}}$$

$$= \frac{E}{\sqrt{R^2 + \omega^2 L^2}} \sin (\omega t - \phi)$$

$$= I \sin (\omega t - \phi)$$

Therefore the complete solution is :

$$i = I \sin (\omega t - \phi) + A \,.\, e^{-R/L\,.\,t}$$

The value of the constant A depends upon the amplitude of the voltage wave at the instant of switching-in.

When this condition is known the value of A may be substituted in the equation, which will then be of a form comparable with equation (61).

DUHAMEL INTEGRAL.

At a time t^{I}, let an e.m.f. of the form $e_{t_{\mathrm{I}}}$ be impressed upon a circuit ; shortly afterwards another e.m.f., $e_{t_{\mathrm{II}}}$ is impressed at time t^{II} ; after that another e.m.f., $e_{t_{\mathrm{III}}}$ is impressed at time t^{III}, and so on.

If the operational solution for the first impressed e.m.f. be : $f(t^{I})$,

then, current due to $e_{tI} = e_{tI}\, f\,(t^{I})$.

current due to $e_{tII} = e_{tII}\, f\,(t^{I} - t^{II})$, etc.

Thus :

$$i = e_{tI}\, f\,(t^{I}) + e_{tII}\, f\,(t^{I} - t^{II}) + e_{tIII}\, f\,(t^{I} - t^{III}) + e_{tIV}\, f\,(t^{I} - t^{IV}),\ \text{etc.}$$

Let the increase in e.m.f. be small and equal to $\delta\, e_I$ and the time interval to $\delta\, t_I$,

$$\text{Then}\quad \delta\, i = e_{tI}\, \delta\, t_I\, f\,(t - t_I)$$

If at time, $t = 0$, e.m.f. $= e_o$, then at this time,

$$\text{current} = e_o\, f\,(t) + \int_o^t e_{tI}\ f\ (t - t_I)\ dt_I$$

If the extreme member of the equation be integrated (by parts) the current solution reduces to a form :

$$i = e_t\, f\,(0) + \int_o^t e\, t_I\ \frac{d}{d\,(t - t_I)} \cdot f\ (t - t_I)\ dt_I$$

This theorem is sometimes known as the superposition theorem.

In conclusion, two examples will be discussed to illustrate the use of the ordinary calculus and the Heaviside's operational calculus.

1 (a). The R.C. circuit (see Fig. 19)

$$R\, i + \frac{1}{C} \int i \,.\, dt = E$$

Differentiating :

$$R\, \frac{di}{dt} + \frac{1}{C}\, i = O$$

$$R\, C\, \frac{di}{dt} + i = O$$

$$R\, C\ di + dt\ i = O$$

$$\frac{di}{i} + \frac{dt}{R\, C} = O$$

Integrating :

$$\log i + \frac{t}{R\, C} + K = O \quad (K = \text{integration constant}).$$

$$\therefore\quad i = e^{-t/RC} \,.\, e^{K} = C\, e^{-t/RC}$$

(C is a new constant equal to the constant e^{K}).

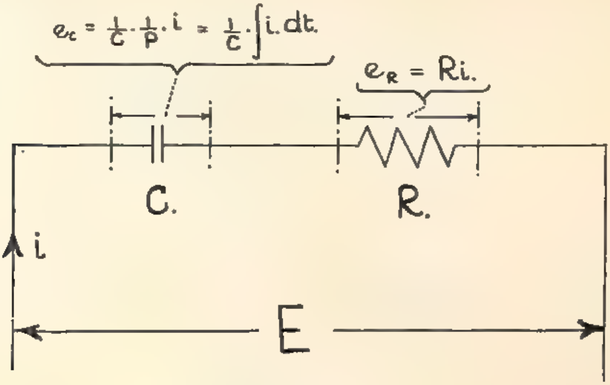


Fig. 19.—R.C. circuit.

At the instant of switching, there is no voltage across the condenser, so that the current is equal to E/R.

$$\therefore \quad i = E/R = C\, e^{+0/RC} = C\, e^{0} = C$$

$$\text{Thus} \quad i = E/R\; e^{-t/RC}$$

(1 b). A similar voltage equation may be written :

$$R\,i + 1/C \,.\, 1/P \,.\, i = E \quad [1]$$

$$i = E \frac{1}{R + 1/C \,.\, 1/P} \quad [1]$$

$$= E \frac{C\,p}{R\,C\,p + 1} \quad [1]$$

Therefore :

$$Y_{(p)} = C\,p$$
$$Y_{(o)} = 0$$
$$Z_{(p)} = R\,C\,p + 1$$
$$Z_{(o)} = 1$$
$$Z'_{(p)} = R\,C$$

To extract the roots, equate $Z_{(p)}$ to zero.

$$\therefore \quad Z_{(p)} = R\,C\,p + 1 = 0$$

$$\therefore \quad p = -\,{}^{1/RC}$$

Substituting in the expansion theorem :

$$i = E\left[\frac{0}{1} + \frac{C\,p\,e^{pt}}{p\,RC}\right]$$

$$= E\left[0 + \frac{C\,(-1/RC)\,e^{-1/RC\,.\,t}}{(-1/RC)\;RC}\right]$$

$$= E\left[\frac{-\,1/R\,e^{-1/RC\,.\,t}}{-\,1}\right]$$

$$= \frac{E}{R}\,e^{-t/RC}$$

(2 a). The R.L. Circuit. See Fig. 20.

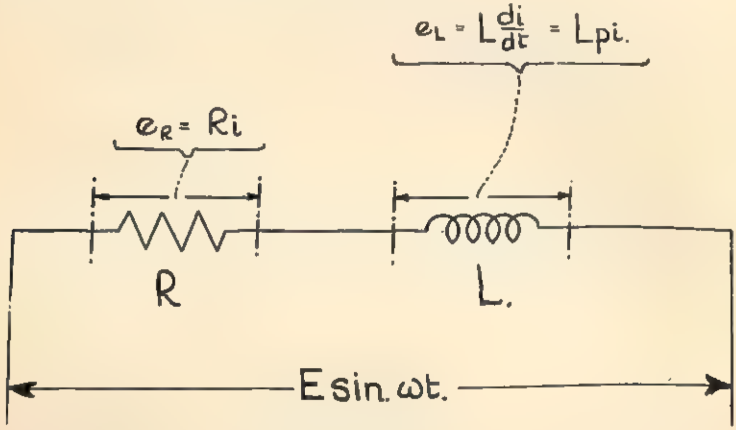


Fig. 20.—R.L. circuit.

The voltage equation is :

$$Ri + L\,\frac{di}{dt} = E \sin \omega t$$

Dividing by L and re-arranging :

$$\frac{di}{dt} + \frac{R}{L}\,i = \frac{E}{L}\,\sin \omega t$$

It will be seen that this is a type of linear equation of the form :

$$\frac{dy}{dx} + p_y = Q$$

Thus :

$$i\, e^{\int R/L \, . dt} = \int \frac{E}{L} \, . \sin \omega t \; e^{\int R/L \, . dt} + K \quad dt$$

Integrating and re-arranging :

$$i = \frac{E}{L} \, e^{-R/L \, . t} \int e^{R/L \, . t} \sin \omega t + K \, . \, e^{-R/L \, . t} \quad dt$$

The term $\int e^{R/L \, . t} \sin . \omega t . dt$ must be integrated by parts.

$\int e^{R/L \, . t} \sin . \omega t . dt$ is comparable with

$$\int u \frac{dv}{dt} \, . \, dt \left(= uv - \int v \frac{du}{dt} \, . \, dt \right)$$

Let $u = \sin \omega t$ $\qquad \dfrac{dv}{dt} \; e^{R/L \, . t}$

$$\frac{du}{dt} = \omega \cos \omega t \qquad v = \frac{L}{R} \, e^{R/L \, . t}$$

$$\int e^{R/L \, . t} \sin \omega t \, dt = \sin \omega t \frac{L}{R} \, e^{R/L \, . t} - \int \frac{L}{R} \, e^{R/L \, . t} \, \omega \cos \omega t \, dt$$

$$= \frac{L}{R} \, e^{R/L \, . t} \sin \omega t - \frac{L}{R} \, \omega \int e^{R/L \, . t} \cos \omega t \, dt$$

Let $u = \cos \omega t$ $\qquad \dfrac{dv}{dt} = e^{R/L \, .}$

$$\frac{du}{dt} = - \omega \sin \omega t$$

$$v = \frac{L}{R} \, e^{R/L \, . t}$$

$$\int e^{R/L \, . t} \cos \omega t \, dt = \cos \omega t \frac{L}{R} \, e^{R/L \, . t} - \int \frac{L}{R} \, e^{R/L \, . t} \, (-\omega \sin \omega t) \, dt$$

$$= \frac{L}{R} \, e^{R/L \, . t} \cos \omega t + \frac{L \omega}{R} \int e^{R/L \, . t} \sin \omega t \, dt$$

$$\therefore \int e^{R/L \, . t} \sin \omega t \, dt = \frac{L}{R} \, e^{R/L \, . t} \sin \omega t - \frac{L \omega}{R} \Big[\cos \omega t \frac{L}{R} \, e^{R/L \, . t}$$

$$+ \frac{L\omega}{R} \int e^{R/L.t} \sin \omega t \, dt \Bigg]$$

$$= \frac{L}{R} e^{R/L.t} \sin \omega t - \frac{L^2\omega}{R^2} e^{R/L.t} \cos \omega t$$

$$- \frac{L^2\omega^2}{R^2} \int e^{R/L.t} \sin \omega t \, dt$$

$$\therefore \int e^{R/L.} \sin \omega t \, dt + \frac{L^2\omega^2}{R^2} \int e^{R/L.t} \sin \omega t \, dt = \frac{L}{R} e^{R/L.t} \sin \omega t$$

$$- \frac{L^2\omega}{R^2} e^{R/L.} \cos \omega t$$

$$\int e^{R/L.} \sin \omega t \, dt \left[1 + \frac{L^2\omega^2}{R^2}\right] = \frac{L}{R} e^{R/L.t} \sin \omega t$$

$$- \frac{L^2\omega}{R^2} e^{R/L.t} \cos \omega t$$

$$\int e^{R/L.t} \sin \omega t \, dt = \frac{e^{R/L.t} \left[\frac{L}{R} \sin \omega t - \frac{L^2\omega}{R^2} \cos \omega t\right]}{\left[1 + \frac{L^2\omega^2}{R^2}\right]}$$

which may be reduced to :

$$\int e^{R/L.t} \sin \omega t \, dt = \frac{e^{R/L.t} \left[\frac{R}{L} \sin \omega t - w \cos \omega t\right]}{\omega^2 + R^2/L^2}$$

Thus :

$$i = \frac{E}{L} \quad \frac{[R/L \, . \sin \omega t - \omega \cos \omega t]}{\omega^2 + R^2/L^2} + K \, . \, e^{-R/L.t}$$

(obviously the terms $e^{-R/L.t}$ and $e^{+R/L.t}$ cancel).

$$i = \frac{E}{L(\omega^2 + R^2/L^2)} \left[\frac{R \sin \omega t - \omega L \cos \omega t}{L}\right] + K e^{-R/L.t}$$

$$= \frac{E}{L^2\omega^2 + R^2} \left[R \sin \omega t - \omega L \cos \omega t\right] + K e^{-R/L.t}$$

$$= \frac{E}{\sqrt{R^2 + L^2\omega^2}} \left[\frac{R \sin \omega t - \omega L \cos \omega t}{\sqrt{R^2 + L^2\omega^2}}\right] + K e^{-R/L.t}$$

From previous examples, it may now be written :

$$i = \frac{E}{\sqrt{R^2 + L^2\omega^2}} \sin(\omega t - \theta) + K e^{-R/L \cdot t}$$

where $\theta = \text{arc} \tan \omega L/R.$

The value of the constant K depends upon the position of the voltage wave at the instant of switching-in. Suppose this to be at a time T.

If t is assumed to be zero, when $i = 0$ and $e = 0$,

$$\left[e = R i + L \frac{di}{dt} = E \sin \omega t \right]$$

then with time from $t = 0$ to $t = T$, at $t = T$.

$$O = \frac{E}{\sqrt{R^2 + L^2\omega^2}} \sin(\omega T - \theta) + K e^{-R/L \cdot t}$$

$$\left(I = \frac{E}{\sqrt{R^2 + L^2\omega^2}} \right)$$

$$\therefore \quad K = - I \sin(\omega T - \theta)\, e^{R/L \cdot t}$$

Finally,

$$i = I \sin(\omega t - \theta) - I e^{-R/L\,(t-T)} \sin(\omega T - \theta)$$

(It is interesting to note that if the instant of switching-in is arranged so that $w T = \theta$, then the transient term is eliminated).

(2b).

$$e = R i + L p i = E \sin \omega t = E e^{jwt} \quad [1]$$

$$p L i + R i = E e^{jwt} \quad [1]$$

$$i = \frac{E e^{jwt}}{p L + R} \quad [1]$$

$$= \frac{E}{p L + R} \cdot \frac{\omega p}{p^2 + w^2}$$

Equating $Z_{(p)} = 0$, Roots $= p_1 = -R/L$

$p_2 = -j\omega$

$p_3 = +j\omega$

$$\therefore \quad i = C_1 e^{-R/L \cdot t} + C_2 e^{-jwt} + C_3 e^{+jwt}$$

From elementary electrical theory,

$$C_2 e^{-jwt} + C_3 e^{+jwt} = \frac{E \sin (\omega t - \theta}{Z}$$

$$Z = \sqrt{R^2 + \omega^2 L^2}$$

$$\theta = \text{arc tan } \omega L/R.$$

$$Y_{(p)} = \omega p : \quad Z_{(p)} = (R + pL)\ (p^2 + \omega^2)$$
$$p = p_1$$

In accordance with previous discussions on products appearing in the denominator, it is permissible to write :

$$p f'_{(p)} F_{(p)} = p L\ (p^2 + \omega^2)$$

$$\therefore \quad -\frac{R}{L} L \left[\frac{R^2}{L^2} + \omega^2 \right] = -R \left[\frac{R^2 + \omega^2 L^2}{L^2} \right]$$

$$Y\ (p = -R/L) = \frac{-\omega R}{L}$$

Thus :

$$\frac{-\omega R/L}{R/L^2\ [R^2 + \omega^2 L^2]} = \frac{-1}{\sqrt{R^2 + \omega^2 L^2}} \cdot \frac{\omega L}{\sqrt{R^2 + \omega^2 L^2}}$$

The complete solution is :

$$i = \frac{E \sin\ (\omega t - \theta)}{Z} - \frac{E}{Z} \sin\ (\omega t - \theta)\ e^{-R/L \cdot t}$$

$$= \frac{E}{Z} \left[\sin\ (\omega t - \theta) - e^{-R/L \cdot t} \sin\ (\omega t - \theta) \right]$$

which is comparable with the previous solution.

CONCLUSION.

Throughout the pamphlet the use of the expansion theorem has been stressed. When one becomes familiar with this branch of operational mathematics it is often found easier to segregate the various functions of p into partial fractions (in a similar manner to that adopted in the discussion of the expansion theorem), and refer to a table of equivalent operators.

In the text, one of these equivalent operators was proved, namely :

$$\frac{1}{p+a}\ [1] = \frac{1}{a}\ [1 - e^{-at}]\ \ [1]$$

Others are :

$$\frac{p}{p+a}\ [1] = e^{-at}\ \ [1]$$

$$\frac{p}{p-a}\ [1] = e^{at}\ \ [1]$$

etc., etc.

As an example of equivalent operators, let the following example be considered :

"Let an inductance of constant value and a resistance of constant value be short-circuited by a switch."

The equation is :

$$R\,i + L\,p\,i = E\sin(\omega t + a)\quad [1]$$

From the section on treatment of alternating currents, the substitution using equivalent operators may be made in one "step," viz. :

$$i = \frac{E}{R+Lp} \cdot \frac{\omega p\cos a + p^2\sin a}{p^2+\omega^2}\ \ [1]$$

etc.

APPENDIX I.

The following list of symbols and definitions may be found useful in conjunction with some of the examples in the pamphlet, and also as an aid for further reading.

M = Bending moment at section considered.

I = Moment of inertia of cross-section pertaining to axis through centre of area of cross-section.

f = Bending stress at height y from centre of area of cross-section(neutral axis).

E = Young's modulus (linear elastic modulus)

$$= \frac{\text{linear stress}}{\text{linear strain}}$$

R = Radius of curvature of beam at section considered.

Resilience of a beam due to bending $= 1/2EI \int M^2\, dx$.

The SHEAR FORCE (SF) at a section of a beam is the algebraic sum of all forces to one side of the section.

The BENDING MOMENT (M) at a section of a beam is the algebraic sum of all moments to one side of the section.

e = instantaneous value of e.m.f. ($= \hat{E} \sin \omega t$).

E_m or $\hat{E}$ = Maximum value of e.m.f.
ω = $2\pi f$ angular velocity (mechanical and electrical).
i = Instantaneous value of current.
I_m or $\hat{I}$ = Maximum value of current.
t = Time in seconds (mechanical and electrical).
E = Root mean square (R.M.S.) value of e.m.f.
I = R.M.S. value of current.
C = Capacitance in farads.
L = Inductance in henries.
R = Resistance in ohms.

$$\text{Reactance} = \left\{ \begin{array}{l} X_C = 1/\omega C \text{ (Capacitive)} \\ X_L = \omega L \text{ (Inductive)} \end{array} \right\}$$

Z = Impedance in ohms

$= \sqrt{R^2 + (X_L - X_c)^2} = \sqrt{R^2 + (\omega L - 1/\omega C)^2}$

$= R \pm jX$

Y = Admittance in mhos (the word 'ohms' reversed).

G = Conductance in mhos $= \gamma \cos \phi$

B = Susceptance in mhos $= \gamma \sin \phi$

Y = $I/E = \sqrt{G^2 + B^2}$

ϕ = Angle of lag of current.

$\pm$ = plus or minus.

$\mp$ = minus or plus.

$\neq$ = is not equal.

$<$ = less than.

$>$ = greater than.

$\gtrless$ = less than or greater than, but not equal.

$\geq$ = equals or is greater than.

$\leq$ = equals or is less than.

$\equiv$ = is identical to.

$\longrightarrow$ = approaches.

$|n|$ = absolute value, modulus or signless value, (*i.e.*, can be + or −).

$\underline{|n}$ or $n!$ = factorial n $(1 . 2 . 3 . 4 . 5 . . . . . n)$.

Greek letters in common use :

α	pronounced	alpha.
β	,,	beta.
γ	,,	gamma.
δ	,,	delta.
ϵ	,,	epsilon.
θ	,,	theta.
κ	,,	kappa.
λ	,,	lambda.
μ	,,	mu.
π	,,	pi (circles only!).
ρ	,,	rho
Σ	,,	sigma.
ϕ	,,	phi.
ψ	,,	psi.
Ω ω	,,	omega

DETERMINANT.

A determinant may be simply expressed, as an arrangement of terms in the form of a square which has a value determined by its development into the factorial quantity of its elements, *i.e.*,

$$\begin{vmatrix} x_1 & y_1 & z_1 & - & - & \mu_1 \\ x_2 & y_2 & z_2 & - & - & \mu_2 \\ x_3 & y_3 & z_3 & - & - & \mu_3 \\ x_4 & y_4 & z_4 & - & - & \mu_4 \\ \vdots & \vdots & \vdots & - & - & \vdots \\ \vdots & \vdots & \vdots & - & - & \vdots \\ x_x & y_x & z_x & - & - & \mu_x \end{vmatrix}$$

is a square array (square formation) of x^2 elements.

The principle element of the above determinant is :

$$x_1,\ y_2,\ z_3 \quad - \quad - \quad \mu_x$$

A minor of an element (say x_3) is the determinant formed of the remaining elements when the column and row containing x_3 are suppressed.

APPENDIX II.

Example.

The following example is taken from "Transients in Electric Circuits," by W. B. Coulthard (Pitman), page 28.

Sweep Circuit for Cathode-ray Oscilloscope.

Introduction.

In most commercial cathode ray oscilloscopes, a sweep or time base circuit is built in, to enable current or voltage v.-time waveforms to be analysed. It is usual to apply the varying voltage to the vertical deflection plates and simultaneously sweep the spot in a horizontal plane. This horizontal base, which is proportional to time, is called the sweep.

One method is by use of a reactance.

It is known that growth of current is in accordance with

$$i = \frac{E}{R}\left[1 - e^{-at}\right]$$

$$= \frac{E}{R}\left[a\,t - \frac{a^2\,t^2}{2\,!} + \ldots\right]$$

As the time is here of the order of microseconds (μ secs), it is seen that

$$i \doteq \left(\frac{E}{R}\right) a\,t = \frac{E\,t}{L}$$

i.e., the sweep deflecting force varies inversely as L.

For a particle of charge e moving at a speed v, the equivalent coil $= e\,v$. If B is the flux density set up by the sweep magnet coil, the force on a particle is $e\,v$ B per unit length.

But force $=$ mass $\times$ acceleration,

and $\quad l = \frac{1}{2}\,a\,t^2$

$$\therefore \quad l = \tfrac{1}{2}\,a\,t^2 = \tfrac{1}{2}\left(\frac{e\,v\,B}{m}\right) t^2$$

$$\text{and} \quad t = \sqrt{\left(\frac{2\,e\,m}{e\,v\,B}\right)} = \frac{i\,L}{E}$$

$$\text{or} \quad l = \left(\frac{L}{E}\right)^2 i^2 \left(\frac{e\,v\,B}{2\,m}\right) c\,m$$

where l is the width of the screen out.

APPENDIX III.

Comparison of Dimensions of Units.

(For use with appropriate Differential Equations).

QUANTITY.	MECHANICAL (Dynamical)	ELECTRICAL (Electro-physical)	HEAT (Thermo-physical)
Length ...	$[L]$	$[L]$	$[L]$
Time, ...	$[T]$	$[T]$	$[T]$
Mass, ...	$[M]$	$[M]$	$[M]$
Force, ...	$[M]\ [L]\ [T]^{-2}$	$[M]\ [L]\ [T]^{-2}$	
Energy (Torque)	$[M]\ [L]^2\ [T]^{-2}$	$[M]\ [L]^2\ [T]^{-2}$	
Quantity of Heat,			$[M]\ [L]^2\ [T]^{-2}$
Power, ...	$[M]\ [L]^2\ [T]^{-3}$	$[M]\ [L]^2\ [T]^{-3}$	
Electric Charge,		$[Q]$	
Velocity, ...	$[L]\ [T]^{-1}$		
Current, ...		$[Q]\ [T]^{-1}$	
Pressure, ...	$[M]\ [L]^{-1}\ [T]^{-2}$		
Momentum, ...	$[M]\ [L]\ [T]^{-1}$		
Gravitational Constant, ...	$[M]^{-1}\ [L]^3\ [T]^{-2}$		
Voltage, ...		$[M]\ [Q]^{-1}\ [L]^2\ [T]^{-2}$	
Resistance, ...		$[M]\ [Q]^{-2}\ [L]^2\ [T]^{-1}$	
Inductance, ...		$[M]\ [Q]^{-2}\ [L]^2$	
Capacitance, ...		$[M]^{-1}\ [Q]^2\ [L]^{-2}\ [T]^2$	
Temperature, ...			θ
Thermal Capacity,			$[L]^2\ [T]^{-2}\ [\theta]^{-1}$
Thermal Conductivity,			$[M]\ [L]\ [T]^{-3}\ [\theta]^{-1}$
Emissivity, ...			$[M]\ [T]^{-3}\ [\theta]^{-1}$
Entropy, ...			$[M]\ [L]^2\ [T]^{-2}\ [\theta]^{-1}$

BIBLIOGRAPHY.

This section serves a dual purpose. Appended hereunder is a list of text-books; reference has been made freely to some of these. The author expresses his appreciation and thanks to the particular authors concerned, and offers these books as a suggested course for further reading.

AITKEN, ... "Determinants and Matrices."
AITKEN, ... "Statistical Mathematics."
ALBERT, ... "Modern Higher Algebra."
BELL, ... "Exponential and Hyperbolic Functions."
BERG, ... "Heaviside's Operational Calculus."
BICKLEY, ... "Application of Engineering Mathematics."
BIRKINSHAW, "Transmission-Line Surges" (*Elec. Review*, July 6th, 1945).
BRUNT, ... "The Combination of Observations."
CAUNT, ... "Introduction to Infinitesimal Calculus."
COULSON, ... "Waves."
COULTHARD, ... "Transients in Electric Circuits."
ESHBACH, ... "Engineering Fundamentals."
FORREST, ... "Calculus for Technical Students."
GIBSON, ... "Elementary Treatise on the Calculus."
GILLESPIE, ... "Integration."
GOLDING and GREEN. "Elementary Practical Mathematics" (Books I., II. and III.)
HYSLOP, ... "Infinite Series"
INCE, ... "Integration of Ordinary Differential Equations."
McCREA, ... "Analytical Geometry of Three Dimensions."
MILLER, ... "Differential Equations."
PHILLIPS, ... "Functions of a Complex Variable."
PIAGGO, ... "Differential Equations."
ROSE, ... "Mathematics for Engineers," Parts I. and II.
RUTHERFORD, "Vector Methods."
SOKOLNIKOFF, "Higher Mathematics for Engineers and Physicists."
STONEY, ... "Calculus for Engineers and Students of Science."
THOMPSON, ... "Calculus Made Easy."
TODHUNTER, ... "Integral Calculus."
TOFT & McKAY, "Practical Mathematics."
TURNBULL, ... "Theory of Equations."
WARREN, ... "Mathematics Applied to Electrical Engineering."
WHITTAKER and ROBINSON, "Calculus of Observations."
WILLIAMSON, "Integral Calculus."

ACKNOWLEDGMENT.

The author wishes to thank W. H. Thompson, Esq., A.M.I.E.E., for much appreciated advice and guidance.

List of A.E.S.D. Printed Pamphlets and Other Publications in Stock.

*The Craft of Tracing, ... * (6d to members, 1/- others,. Post free).	
Lost Volumes from Cones, ...	W. R. Thomson
The Parallel Operation of Diesel-Alternator Sets, ...	G. H. Bradbury
Magnetism in Industry and Science, ...	H. E. Hutter
Brakes for Locomotives and Other Rail Units, ...	George W. McArd
Ward Leonard Control of Planer Type Machine Tools,	B. Feldbauer
Surface Condenser Design, ...	Robert B. Neilson
Reinforced Concrete, ...	J. McHardy Young
Dynamic Balancing, ...	A. Bell
Fluid Film Lubricated Bearing Practice, ...	G. S. Elfer
*Power Station Civil Engineering and Building Works, * (1/6 to members, 3/- others. Post free).	T. H. Carr
The Current Rating of Paper Insulated Cables, ...	C. C. Barnes
The Design of Structural Welded Details, ...	Frank H. Abrahams
Machines for Dynamic Balancing, ...	Ralph M. Eggleston
An Introduction to the Construction of Nomographs,	William W. Calder
Foundations, ...	J. McHardy Young
An Introduction to the Cathode Ray Oscillograph, ...	F. G. Spreadbury
Panels and Panelling, ...	K. E. Harper
Design of Concrete Sections, ...	W. Scott Wilson
Industrial Ventilation and Air Treatment, ...	M. R. Morton
Some Notes on the Stresses in Flat Rectangular Plates Under Pressure and The Design of Hopper and Suspended Type Bunkers, ...	George D. Mouat
Electricity Sub-Stations, ...	T. H. Carr.
The Analysis of Braced Frames by the Method of Shear Coefficients, ...	F. H. Abrahams
Concrete Slab and Girder Floors, ...	W. Scott Wilson
Helical Gears for Marine Turbines, ...	Duncan Carmichael
An Elementary Introduction to Industrial Electronics,	Kenneth J. Field
*Design of Transformers, ... * (1/6 to members, 3/- others. Post Free).	T. Waterhouse
Production of Accurate Screw Threads, ...	J. W. K. Murch
High Pressure Hydraulic Control Equipment,	R. Hartkopf & J. Rodger
The Fundamentals of Metallurgy, ...	Kenneth G. Keeling
Mechanical Methods of Calculation, ...	Sydney Hartley-Smith
Turbo-Supercharging of Internal Combustion Engines,	C. V. J. Mackenzie-Kennedy
Furnace Brickwork, ...	H. D. Hendrick
Electrical Protective Equipment, ...	T. H. Carr
Concrete Columns and Founds, ...	W. Scott Wilson
Methods of Approximate Integration, ...	George D. Mouat
Design Examples for Young Draughtsmen (Reprint),	J. W. Hamilton
*An Introduction to Engineering Mathematics, with Reference to Heaviside's Operational Calculus, * (1/6 to members, 3/- others. Post free).	Ronald J. Birkinshaw
Electronic Heating, ...	Geo. R. Cooper

All not otherwise marked are 1/- to members, 2/- others, post free.

List of A.E.S.D. Data Sheets.

1. Safe Load on Machine-Cut Spur Gears.
2. Deflection of Shafts and Beams
3. Deflection of Shafts and Beams (Instruction Sheet) } Connected.
4. Steam Radiation Heating Chart.
5. Horse-Power of Leather Belts, etc.
6. Automobile Brakes (Axle Brakes)
7. Automobile Brakes (Transmission Brakes) } Connected.
8. Capacities of Bucket Elevators.
9. Valley Angle Chart for Hoppers and Chutes.
10. Shafts up to 5½-in. diameter, subjected to Twisting and Combined Bending and Twisting.
11. Shafts, 5¾ to 26-inch diameter, subjected to Twisting and Combined Bending and Twisting.
12 Ship Derrick Booms.
13. Spiral Springs (Diameter of Rd. or Sq. Wire)
14. Spiral Springs (Compression).
15. Automobile Clutches (Cone Clutches).
16. " " (Plate Clutches).
17. Coil Friction for Belts, etc.
18. Internal Expanding Brakes. Self-Balancing Brake Shoes (Force Diagram)
19. Internal Expanding Brakes. Angular Proportions for Self-Balancing } Connected.
20. Referred Mean Pressure Cut-Off, etc.
21. Particulars for Balata Belt Drives.
22. 7/8" Square Duralumin Tubes as Struts.
23. 1" " " " "
24. 3/4" Sq. Steel Tubes as Struts 30 ton yield).
25. 7/8" " " " (30 ton yield).
26. 1" " " " (30 ton yield).
27. 3/4" " " " (40 ton yield).
28. 7/8" " " " (40 ton yield).
29. 1" " " " (40 ton yield).
30. Moments of Inertia of Built-up Sections (Tables)
31. Moments of Inertia of Built-up Sections (Instructions and Examples) } Connected.
32. Reinforced Concrete Slabs (Line Chart)
33. Reinforced Concrete Slabs (Instructions and Examples) } Connected.
34. Capacity and Speed Chart for Troughed Band Conveyors.
35. Screw Propeller Design (Sheet 1, Diameter Chart)
36. " " " (Sheet 2, Pitch Chart) } Connected.
37. " " " (Sheet 3, Notes and Examples)
38. Open Coil Conical Springs.
39. Close Coil Conical Springs.
40. Trajectory Described by Belt Conveyors.
41. Metric Equivalents.
42. Useful Conversion Factors.
43. Torsion of Non-Circular Shafts.
44. Railway Vehicles on Curves.
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